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The Character of Pure Spinors

Nathan Berkovits[1] and Nikita A. Nekrasov[2†]

[1] *Instituto de Física Teórica, Universidade Estadual Paulista, Saõ Paulo, SP 01405, Brasil*

[2] *Institute des Hautes Etudes Scientifiques, Le Bois-Marie, Bures-sur-Yvette, F-91440 France*

The character of holomorphic functions on the space of pure spinors in ten, eleven and twelve dimensions is calculated. From this character formula, we derive in a manifestly covariant way various central charges which appear in the pure spinor formalism for the superstring. We also derive in a simple way the zero momentum cohomology of the pure spinor BRST operator for the $D = 10$ and $D = 11$ superparticle.

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† On leave of absence from: Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia

1. Introduction

Five years ago, a new formalism was developed for covariantly quantizing the superstring [1]. This formalism has the new feature that the worldsheet ghosts are ten-dimensional bosonic spinors λ^α which satisfy the pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ where α=1 to 16 and $m = 0$ to 9. Because of this pure spinor constraint on the worldsheeet ghost, the worldsheet antighost w_α is only defined up to the gauge transformation $\delta w_\alpha = \Lambda_m (\gamma^m_{\alpha\beta} \lambda^\beta)$ for arbitrary gauge parameter Λ_m. This implies that the worldsheet antighost can only appear in the combinations

$$N_{mn} = \tfrac{1}{2} \lambda^\beta (\gamma_{mn})_\beta^\alpha w_\alpha \qquad \text{and} \qquad J = \lambda^\alpha w_\alpha,$$

which are identified with the Lorentz current and ghost-number current.

To compute scattering amplitudes, one needs to know the OPE's of N_{mn} and J with themselves and with the stress tensor. Although the single poles in these OPE's are easily determined from symmetry properties, the central charges need to be explicitly computed. One method for computing these central charges is to non-covariantly solve the pure spinor constraint and express N_{mn} and J in terms of unconstrained variables. Although this method is not manifestly covariant, one can easily verify that the result for the central charges is Lorentz covariant.

Nevertheless, it would be useful to have a manifestly Lorentz-covariant method for computing these central charges. In addition to the desire to manifestly preserve as many symmetries as possible, Lorentz-covariant methods may be necessary if one wants to generalize the OPE computations in a curved target-space background.

In this paper, it will be shown how to compute the central charges in a manifestly Lorentz-covariant manner. By describing the space of pure spinors as the cone over the $SO(10)/U(5)$ coset and computing the characters of holomorphic functions on this coset, it will be possible to reinterpret the constrained λ^α as an infinite set of unconstrained ghosts-for-ghosts. The central charges in the OPE's can then be computed by summing the contributions to the central charges from the infinite set of ghosts-for-ghosts. This procedure is related to Chesterman's approach to pure spinors [2], and allows one to complete his construction of a "second" BRST operator. Replacing a constrained pure spinor with unconstrained ghosts-for-ghosts has also been considered in [3], where a BRST operator was constructed from only first-class constraints as in the programme of [4].

In addition to being useful in ten dimensions for covariantly quantizing the superstring, pure spinors are also useful in eleven dimensions for covariantly quantizing the $D = 11$ superparticle and, perhaps, the supermembrane [5]. Our covariant methods can be easily generalized to compute characters of functions of pure spinors in any dimension, in particular eleven and twelve. Although the Lorentz central charge does not have any obvious meaning when $D = 11$, the ghost number anomaly does have a physical meaning and is related to the construction of a BV action [5] and superparticle scattering amplitudes [6]in $D = 11$.

In eleven dimensions, there are two possible definitions of pure spinors which are either

$$\lambda\gamma^I\lambda = 0 \qquad \text{or} \qquad \lambda\gamma^{IJ}\lambda = \lambda\gamma^I\lambda = 0,$$

where $I, J = 0$ to 10 and λ has 32 components. The first definition was used for covariantly quantizing the $D = 11$ superparticle and gives rise to a BRST description of linearized $D = 11$ supergravity [5][7]. The second definition was used by Howe in [8] and can be obtained by dimensionally reducing a $D = 12$ pure spinor which satisifes $\lambda\gamma^{MN}\lambda = 0$ for $M, N = 0$ to 11 where the chiral twelve dimensional spinor λ has again 32 components. We explicitly compute the characters using both definitions of $D = 11$ pure spinors, however, we have been unable to give a physical interpretation to characters coming from the second definition.

In section 2 of this paper, non-covariant methods will be used to solve the pure spinor constraint in any even dimension and compute the central charges. In D Euclidean dimensions, these methods manifestly preserve $U(D/2)$ invariance. One method requires bosonization and is a generalization of the parameterization used in [1] for $D = 10$ pure spinors. An alternative method uses a new parameterization of pure spinors which does not require bosonization.

In section 3, the character of holomorphic functions on an $SO(10)/U(5)$ coset will be computed using two different covariant methods. This character formula will then be related to the zero momentum BRST cohomology of a $D = 10$ superparticle which describes super-Yang-Mills. Moreover, it will be shown how to generalize the character formula to describe superparticle states with non-zero momentum.

In section 4, the character formula will be used to reinterpret the pure spinor as an infinite set of unconstrained ghosts-for-ghosts, and the central charges will then be computed by summing the contributions from the unconstrained ghosts-for-ghosts.

In section 5, we repeat this covariant procedure using a $D=11$ pure spinor satisfying $\lambda\gamma^I\lambda=0$ where $I=0$ to 10. The character of functions on this space of pure spinors will be related to the BRST cohomology of a $D=11$ superparticle which describes linearized $D=11$ supergravity. Furthermore, central charges associated with this $D=11$ pure spinor will be covariantly computed by reinterpreting the pure spinor as an infinite set of unconstrained ghosts-for-ghosts.

In section 6, we shall compute the character of functions of a $D=11$ pure spinor satisfying $\lambda\gamma^I\lambda=0$ as well as $\lambda\gamma^{IJ}\lambda=0$, which is equivalent to computing the character of functions of a $D=12$ pure spinor satisfying $\lambda\gamma^M\lambda=0$ where $M=0$ to 11. In this case, the BRST cohomology associated with the character formula is complicated and we have not yet found a physical interpretation for it. We do see, however, an indication for the presence of a self-dual $D=12$ six-form (field strength?) in the BRST spectrum, which dimensionally reduces to a $D=11$ five-form.

2. Non-Covariant Computation of Central Charges

In this section, the central charges in pure spinor OPE's will be computed by solving, non-covariantly, the pure spinor constraint in any even dimension. The computations are quite simple and will be performed using two alternative parameterizations of a pure spinor. The first parameterization requires bosonization of one of the worldsheet fields, while the second parameterization does not. We will later rederive these results using manifestly covariant computations.

In $D=2d$ Euclidean dimensions, a pure spinor λ^α is constrained to satisfy $\lambda^\alpha(\sigma^{m_1..m_j})_{\alpha\beta}\lambda^\beta=0$ for $0\le j<d$, where $m=1$ to $2d$, $\alpha=1$ to 2^{d-1}, and $\sigma^{m_1...m_j}_{\alpha\beta}$ is the antisymmetrized product of j Pauli matrices. This implies that $\lambda^\alpha\lambda^\beta$ can be written as

$$\lambda^\alpha\lambda^\beta=\frac{1}{n!\,2^d}\sigma^{\alpha\beta}_{m_1...m_d}\,(\lambda^\gamma\sigma^{m_1...m_d}_{\gamma\delta}\lambda^\delta) \tag{2.1}$$

where $\lambda\sigma^{m_1...m_d}\lambda$ defines an d-dimensional complex hyperplane. This d-dimensional complex hyperplane is preserved up to a phase by a $U(d)$ subgroup of $SO(2d)$ rotations. So projective pure spinors in $D=2d$ Euclidean dimensions parameterize the coset space $SO(2d)/U(d)$, which implies that λ^α has $(d^2-d+2)/2$ independent complex degrees of freedom [9].

2.1. Method with bosonization

One method for computing central charges is to non-covariantly solve the pure spinor constraint of (2.1) and express λ^α in terms of the $(d^2-d+2)/2$ independent degrees of freedom. This was done in [1] by decomposing the 2^{d-1} components of λ^α into $SU(d)\times U(1)$ representations as

$$(\lambda^{\frac{d}{2}} = \gamma, \quad \lambda^{\frac{d-4}{2}}_{[ab]} = \gamma u_{[ab]}, \quad \lambda^{\frac{d-8}{2}}_{[abcd]} = -\frac{1}{8}\gamma u_{[ab}u_{cd]}, \quad \lambda^{\frac{d-12}{2}}_{[abcdef]} = -\frac{1}{48}\gamma u_{[ab}u_{cd}u_{ef]}, \quad ...) \tag{2.2}$$

where the superscript on λ is the $U(1)$ charge, γ is an $SU(d)$ scalar with $U(1)$ charge $\frac{d}{2}$, and u_{ab} is an $SU(d)$ antisymmetric two-form with $U(1)$ charge -2.

Using this decomposition, the naive ghost-number current is $J=\gamma\beta$ and the naive $U(1)$ Lorentz current is $N_{U(1)} = \frac{d}{2}\gamma\beta - u_{ab}v^{ab}$ where β and v^{ab} are the conjugate momenta for γ and u_{ab}. However, this naive definition would imply a double pole in the OPE of J and $N_{U(1)}$ which would violate Lorentz invariance. As shown in [1], this double pole can be avoided by bosonizing the (β,γ) fields as $(\beta = \partial\xi e^{-\phi}, \gamma = \eta e^{\phi})$ and including terms proportional to $(\partial\phi + \eta\xi)$ in J and $N_{U(1)}$. These terms do not modify the OPE's of N and J with λ^α since $(\partial\phi+\eta\xi)$ has no poles with λ^α. However, they do contribute to the central charges which can be understood as normal-ordering contributions. The coefficients of these normal-ordering contributions can be easily determined by requiring that N_{mn} is a conformal primary field which has no singularities with J.

The formulas for the currents in $D=2d$ are a generalization of the $D=10$ formulas in [1] and are given by

$$J = -\frac{5}{2}\partial\phi - \frac{3}{2}\eta\xi, \tag{2.3}$$

$$N^{ab} = v^{ab},$$

$$N_a^b = -u_{ac}v^{bc} + \delta_a^b(\frac{5}{4}\eta\xi + \frac{3}{4}\partial\phi),$$

$$N_{ab} = (d-2)\partial u_{ab} + u_{ac}u_{bd}v^{cd} + u_{ab}(\frac{5}{2}\eta\xi + \frac{3}{2}\partial\phi),$$

$$T = \tfrac{1}{2}v^{ab}\partial u_{ab} - \tfrac{1}{2}\partial\phi\partial\phi - \eta\partial\xi + \tfrac{1}{2}\partial(\eta\xi) + (1-d)\partial(\partial\phi+\eta\xi),$$

where T is the stress tensor and the worldsheet fields satisfy the OPE's

$$\eta(y)\xi(z) \to (y-z)^{-1}, \quad \phi(y)\phi(z) \to -\log(y-z), \quad v^{ab}u_{cd} \to \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}. \tag{2.4}$$

2.2. Method without bosonization

One can avoid the need for bosonization by using an alternative $U(d)$-covariant decomposition of λ^α in which the pure spinor is parameterized by an $SU(d)$ vector e^a and an $SU(d)$ antisymmetric two-form g^{ab}. Since a vector and two-form describe $(d^2+d)/2$ degrees of freedom and a pure spinor contains only (d^2-d+2) degrees of freedom, this parameterization will contain gauge invariances.

In this decompostion, λ^α splits into the $SU(d)\times U(1)$ representations

$$(\lambda^a_{\frac{2-d}{2}} = e^a, \quad \lambda^{[abc]}_{\frac{6-d}{2}} = \tfrac{1}{2}e^{[a}g^{bc]}, \quad \lambda^{[abcde]}_{\frac{10-d}{2}} = -\frac{1}{8}e^{[a}g^{bc}g^{de]}, \quad ...) \tag{2.5}$$

where the subscript on λ is the $U(1)$ charge, e^a is an $SU(d)$ vector with $U(1)$ charge $\frac{2-d}{2}$, and g^{ab} is an $SU(d)$ antisymmetric two-form with $U(1)$ charge $+2$. Note that when d is odd, the $U(d)$ representations in (2.2) and (2.5) are the same (although they are written in opposite order). But when d is even, the $U(d)$ representations in (2.2) and (2.5) are different since the pure spinors λ^α in (2.2) and (2.5) have opposite chirality.

Using the decomposition of (2.5), the parameterization of λ^α is invariant under the gauge transformation

$$\delta g^{ab} = \Omega^{[a}e^{b]}, \tag{2.6}$$

which in turn has the gauge-for-gauge invariance,

$$\delta\Omega^a = \Lambda e^a. \tag{2.7}$$

Gauge-fixing the invariance parameterized by Ω^a introduces a fermionic ghost ψ^a which transforms as an $SU(d)$ vector with $U(1)$ charge $\frac{d+2}{2}$ and ghost-number -1. (Note that λ^α has ghost-number one, so e^a has ghost-number one and g^{ab} has ghost-number zero.) And fixing the gauge-for-gauge invariance parameterized by Λ introduces a bosonic ghost-for-ghost r which transforms as an $SU(d)$ scalar with $U(1)$ charge d and ghost-number -2.

In terms of these fields and their conjugate momenta,

$$(e^a, f_a; \quad g^{ab}, h_{ab}, \quad \psi^a, \eta_a; \quad r, s)$$

the worldsheet action is

$$S = \int d^2z (f_a \bar\partial e^a + \tfrac{1}{2}h_{ab}\bar\partial g^{ab} - \eta_a\bar\partial\psi^a + s\bar\partial r),$$

the stress tensor is

$$T = f_a \partial e^a + \tfrac{1}{2} h_{ab} \partial g^{ab} - \eta_a \partial \psi^a + s \partial r,$$

the ghost-number current is

$$J = e^a f_a - \psi^a \eta_a - 2rs, \tag{2.8}$$

and the Lorentz currents are

$$N_{ab} = h_{ab},$$

$$N^a_b = e^a f_b + g^{ac} h_{bc} + \psi^a \eta_b + \delta^a_b (-\tfrac{1}{2} e^c f_c + \tfrac{1}{2} \psi^c \eta_c + rs),$$

$$N^{ab} = (d-2)\partial g^{ab} - \tfrac{1}{2} g^{[ab} N^{c]}_c + g^{ab}(g^{cd} h_{cd}) - g^{ad} g^{be} h_{de}.$$

Note that $N^{ab} = (d-2)\partial g^{ab} - \frac{1}{2} g^{[ab} \hat{N}^{c]}_c + g^{ad} g^{be} h_{de}$ where $\hat{N}^a_b = N^a_b - g^{ac} h_{bc}$.

2.3. Central charges

Using either of the two parameterizations of a pure spinor, the OPE's of the currents in (2.3) or (2.8) can be computed to be

$$N_{mn}(y)\lambda^\alpha(z) \sim \tfrac{1}{2} \frac{1}{y-z} (\gamma_{mn}\lambda)^\alpha, \quad J(y)\lambda^\alpha(z) \sim \frac{1}{y-z} \lambda^\alpha, \tag{2.9}$$

$$N^{kl}(y) N^{mn}(z) \sim \frac{2-d}{(y-z)^2} (\eta^{n[k} \eta^{l]m}) + \frac{1}{y-z} (\eta^{m[l} N^{k]n} - \eta^{n[l} N^{k]m}),$$

$$J(y)J(z) \sim -\frac{4}{(y-z)^2}, \quad J(y) N^{mn}(z) \sim 0,$$

$$N_{mn}(y) T(z) \sim \frac{1}{(y-z)^2} N_{mn}(z), \quad J(y) T(z) \sim \frac{2-2d}{(y-z)^3} + \frac{1}{(y-z)^2} J(z),$$

$$T(y)T(z) \sim \tfrac{1}{2} \frac{d(d-1)+2}{(y-z)^4} + \frac{2}{(y-z)^2} T(z) + \frac{1}{y-z} \partial T.$$

So the conformal central charge is $d(d-1)+2$, the ghost-number anomaly is $2-2d$, the Lorentz central charge is $(2-d)$, and the ghost-number central charge is -4. One can verify the consistency of these charges by considering the Sugawara construction of the stress tensor

$$T = \frac{1}{2(k+h)} N_{mn} N^{mn} + \frac{1}{8} JJ + \frac{d-1}{4} \partial J \tag{2.10}$$

where k is the Lorentz central charge, h is the dual Coxeter number for $SO(2d)$, and the coefficient of ∂J has been chosen to give the ghost-number anomaly $2-2d$. Setting $k = 2-d$, one finds that the $SO(2d)$ current algebra contributes $(2d-1)(2-d)$ to the conformal central charge and the ghost current contributes $1 + 3(d-1)^2$ to the conformal central charge. So the total conformal central charge is $d(d-1)+2$ as expected.

3. The Character of Ten-dimensional Pure Spinors

In this section, we shall derive a character formula for pure spinors which will later be used to covariantly compute the central charges.

3.1. Pure spinors and Q operator

Consider the space $\mathcal{F}$ of holomorphic functions $\Psi(\lambda, \theta)$, where $\lambda = (\lambda^\alpha) \in \mathbf{S}_-$, $\theta = (\theta^\alpha) \in \mathbf{S}_-$ are the ten dimensional chiral spinors; λ is bosonic and θ is fermionic. In addition, λ obeys the pure spinor constraint $\lambda\gamma^m\lambda = 0$. The space of solutions to this constraint, the space of pure spinors, will be denoted by $\mathcal{M}_{10}$. The space $\mathcal{F}$ can be described as:

$$\mathcal{F} = \mathbf{C}\left[\mathcal{M}_{10}\right] \otimes \Lambda^\bullet \mathbf{S}_- \tag{3.1}$$

and is acted upon by the fermionic nilpotent operator

$$Q = \lambda^\alpha \frac{\partial}{\partial\theta^\alpha}. \tag{3.2}$$

This operator commutes with the action of the ghost charge operator

$$K = \lambda\frac{\partial}{\partial\lambda} + \theta\frac{\partial}{\partial\theta} \tag{3.3}$$

and the Lorentz, $Spin(10)$ generators:

$$M_{mn} = \frac{1}{2}\lambda\gamma_{mn}\frac{\partial}{\partial\lambda} + \frac{1}{2}\theta\gamma_{mn}\frac{\partial}{\partial\theta}. \tag{3.4}$$

As a consequence, the ghost charge and the Lorentz group act on the space $\mathcal{H}$ of Q-cohomology. Note that K and M_{mn} differ from the ghost and Lorentz charges of section 2 since they contain θ dependence. The operator Q lowers the fermionic charge F,

$$F = \theta\frac{\partial}{\partial\theta}, \tag{3.5}$$

by one unit. The spaces $\mathcal{H}, \mathcal{F}$ are bi-graded by the ghost charge and by the fermionic charge (the number of θ's):

$$\begin{gathered} \mathcal{H} = \bigoplus_{p,q} \mathcal{H}^{p,q}\ , \qquad \mathcal{F} = \bigoplus_{p,q} \mathcal{F}^{p,q} \\ p = K \qquad\qquad q = 16 - F \end{gathered} \tag{3.6}$$

We want to calculate the character of $\mathcal{H}$:

$$\chi(t,\sigma) = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^q \ t^K \ \mathbf{g}\right] \tag{3.7}$$

where

$$\mathbf{g} = \begin{pmatrix} \mathbf{L}_1 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{L}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{L}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{L}_4 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{L}_5 \end{pmatrix} = \exp i\sigma\cdot M \ , \tag{3.8}$$

where

$$\mathbf{L}_a = \begin{pmatrix} \cos\,\sigma_a & \sin\,\sigma_a \\ -\sin\,\sigma_a & \cos\,\sigma_a \end{pmatrix}, \qquad a = 1,\ldots 5, \tag{3.9}$$

and $M = (M_{mn})$, $m,n = 1,\ldots 10$, are the generators of $Spin(10)$. This character contains information about the spin content of the Q cohomology. In fact the knowledge of the character gives the full spin content.

For future use we introduce the notation:

$$\chi_p = \sum_{q=0}^{16}(-1)^q \mathrm{Tr}_{\mathcal{H}^{p,q}}(\mathbf{g}). \tag{3.10}$$

3.2. Relating χ to the character of pure spinors

We claim that χ can be calculated without knowing the cohomology of Q. The idea is the same as in the calculation of the Euler characteristic of a manifold. One can do it knowing the Betti numbers, but one can also do it from using any cell decomposition of the manifold, without actually computing its cohomology.

Moreover, we claim:

$$\chi_p = \sum_{q=0}^{16}(-1)^q \mathrm{Tr}_{\mathcal{F}^{p,q}}(\mathbf{g}). \tag{3.11}$$

This is proven in a standard way:

$$\begin{aligned} \mathrm{Tr}_{\mathcal{F}^{p,q}} =& \mathrm{Tr}_{\mathcal{Z}^{p,q}} + \mathrm{Tr}_{\mathcal{F}^{p,q}/\mathcal{Z}^{p,q}} = \\ & \mathrm{Tr}_{\mathcal{H}^{p,q}} + \mathrm{Tr}_{\mathcal{B}^{p,q}} + \mathrm{Tr}_{\mathcal{B}^{p,q+1}} \end{aligned} \tag{3.12}$$

where $\mathcal{Z} = \mathrm{Ker}Q$, $\mathcal{B} = \mathrm{Im}Q$, and (3.12) shows that the contribution of $\mathcal{B}$ drops out from the alternating sum in (3.11).

Now the problem of calculating χ is much simpler. In fact, $\Lambda^\bullet \mathbf{S}_-$ contributes

$$\mathrm{Det}_{\mathbf{S}_-}\left(1 - t\mathbf{g}\right) = \prod_{\text{even } \# \text{ of } -} \left(1 - t\ e^{\frac{i}{2}(\pm\sigma_1 \pm\sigma_2 \pm \ldots \pm\sigma_5)}\right) \tag{3.13}$$

and it remains to calculate the character of $\mathbf{g}, t$ acting on the space $\mathbb{H} = \mathbf{C}\left[\mathcal{M}_{10}\right]$ of all holomorphic functions on the space of pure spinors:

$$Z_{10}(t,\sigma) \equiv \mathrm{Tr}_{\mathbb{H}}\left[t^K e^{i\sigma\cdot M}\right]$$

The answer, which will be derived in the following subsections, is:

$$\boxed{Z_{10}(t,\sigma) = \frac{\left(1 - t^2 V + t^3 S_+ - t^5 S_- + t^6 V - t^8\right)}{\prod_{\text{even } \# \text{ of } -}\left(1 - t\ \exp\frac{i}{2}\left(\pm\sigma_1 \pm\sigma_2 \ldots \pm\sigma_5\right)\right)}} \tag{3.14}$$

where (the notations are obvious):

$$\begin{aligned} V &= \mathrm{Tr}_{\mathbf{V}}\left[\mathbf{g}\right] = \sum_{a=1}^{5} 2\ \cos\,\sigma_a \\ S_+ &= \mathrm{Tr}_{\mathbf{S}_+}\left[\mathbf{g}\right] = \sum_{\text{even } \# \text{ of } +} \exp\frac{i}{2}\left(\pm\sigma_1 \pm\sigma_2 \ldots \pm\sigma_5\right) \\ S_- &= \mathrm{Tr}_{\mathbf{S}_-}\left[\mathbf{g}\right] = \sum_{\text{even } \# \text{ of } -} \exp\frac{i}{2}\left(\pm\sigma_1 \pm\sigma_2 \ldots \pm\sigma_5\right) \end{aligned} \tag{3.15}$$

In the following two subsections, we shall first derive the formula of (3.14) using a reducibility method and will then rederive it using a fixed-point method. Although the reducibility method is relatively simple for $D = 10$ and is related to the zero-momentum spectrum of the superparticle, it becomes more complicated for $D > 10$. So for higher-dimensional pure spinors, it will be simpler to use the fixed-point method. We shall then use the formula of (3.14) to compute various central charges in a manifestly covariant manner.

3.3. Reducibility method

To derive the formula of (3.14) using the reducibility method, note that $Z_{10}(t,\sigma)$ can be written as $P_{10}(t,\sigma)/Q_{10}(t,\sigma)$ where $(Q_{10}(t,\sigma))^{-1}$ is the partition function for an unconstrained sixteen-component chiral spinor. Furthermore, it will now be shown that the numerator

$$P(t,\sigma) = 1 - t^2 V + t^3 S_+ - t^5 S_- + t^6 V - t^8 \tag{3.16}$$

comes from the constraints implied by the pure spinor condition $\lambda\gamma^m\lambda = 0$.

As discussed in the appendix of [10], the reducibility conditions for the pure spinor constraint $\lambda\gamma^m\lambda = 0$ can be described by the nilpotent operator

$$\widetilde{Q} = (\widetilde{\lambda}\gamma^m\widetilde{\lambda})b_m + c^m(\widetilde{\lambda}\gamma_m f) + (\widetilde{\lambda}\gamma_m\widetilde{\lambda})(j\gamma^m g) - 2(j_\alpha\widetilde{\lambda}^\alpha)(g_\beta\widetilde{\lambda}^\beta) \tag{3.17}$$

$$+(k\gamma_m\widetilde{\lambda})r^m + (\widetilde{\lambda}\gamma^m\widetilde{\lambda})s_m t,$$

where $\widetilde{\lambda}^\alpha$ is an unconstrained spinor and (c^m, b_m), (g_α, f^α), (k^α, j_α), (s_m, r^m), and (u, t) are pairs of variables and their conjugate momentum which have been added to the Hilbert space. In order that $\widetilde{Q}$ is fermionic of ghost-number zero, the pairs (c^m, b_m), (k^α, j_α) and (u, t) are defined to be fermions of ghost-number $(2, -2)$, $(5, -5)$ and $(8, -8)$ respectively, and the pairs (g_α, f^α) and (s_m, r^m) are defined to be bosons of ghost-number $(3, -3)$ and $(6, -6)$ respectively.[1] One can check that the coupling of the variables (g_α, f^α), (k^α, j_α), (s_m, r^m), and (u, t) in (3.17) describes the reducibility conditions satisfied by $\widetilde{\lambda}\gamma^m\widetilde{\lambda} = 0$. For example,

$$(\widetilde{\lambda}\gamma^m\widetilde{\lambda})(\gamma_m\widetilde{\lambda})_\alpha = 0 \tag{3.18}$$

implies the coupling to f^α in (3.17),

$$((\widetilde{\lambda}\gamma^n\widetilde{\lambda})\gamma_n^{\alpha\beta} - 2\lambda^\alpha\lambda^\beta)(\gamma_m\widetilde{\lambda})_\alpha = 0 \tag{3.19}$$

implies the coupling to j_α in (3.17),

$$(\widetilde{\lambda}\gamma_m)_\beta((\widetilde{\lambda}\gamma^n\widetilde{\lambda})\gamma_n^{\alpha\beta} - 2\lambda^\alpha\lambda^\beta) = 0 \tag{3.20}$$

implies the coupling to r_m in (3.17), and

$$(\widetilde{\lambda}\gamma^m\widetilde{\lambda})(\widetilde{\lambda}\gamma_m)_\beta = 0 \tag{3.21}$$

implies the coupling to t in (3.17).

By comparing (3.17) with (3.16), one can easily check that each term in $P(t, \sigma)$ corresponds to a reducibility condition. For example, the term $-t^2 V$ corresponds to the original

[1] In the appendix of [10], $\widetilde{Q}$ was defined to carry ghost-number one, which implied different ghost numbers for the variables. In the context of this paper, $\widetilde{Q}$ is defined to carry ghost-number zero with respect to the ghost charge K of (3.3). Therefore, the t dependence in (3.16) counts the ghost number.

pure spinor constraint, the term $t^3 S_+$ corresponds to the reducibility condition of (3.18), the term $-t^5 S_-$ corresponds to the reducibility condition of (3.19), etc. So as claimed, $Z_{10}(t,\sigma) = P_{10}(t,\sigma)/Q_{10}(t,\sigma)$ correctly computes the character formula.

As discussed in [10], the operator of (3.17) is related to the zero momentum spectrum of the $D = 10$ superparticle. It was shown that the super-Yang-Mills ghost, gluon, gluino, antigluino, antigluon and antighost are described by the states $(1, c_m, k^\alpha, g_\alpha, s_m, u)$ which appear in (3.17).[2] The fact that these states describe the zero momentum spectrum can be easily seen by computing the cohomology of (3.2). Since the contribution from the partition function for θ cancels the denominator of (3.14), the cohomology of (3.2) is described by the numerator $P(t,\vec{\sigma})$ of (3.16).

3.4. The fixed point formula

In this section we discuss the spaces of pure spinors $\mathcal{M}_D$ in various dimensions.

In the case of even dimension $D = 2d$, one can use the method of calculating the character which is well-known in representation theory and is essentially due to H. Weyl. We make use of the fact that the space of pure spinors $\mathcal{M}_{2d}$ is a complex cone over a compact projective variety $\mathcal{X}_d = \mathbf{P}(\mathcal{M}_{2d})$. The space $\mathbb{H}_d$ of holomorphic functions on $\mathcal{M}_{2d}$ can be then identified with the space

$$\mathbb{H}_d = \bigoplus_{n=0}^{\infty} \mathbf{H}^0\left(\mathcal{X}_d, \mathcal{O}(n)\right) \tag{3.22}$$

of holomorphic sections of all powers of the line bundle $\mathcal{L} = \mathcal{O}(1)$, whose total space is the space $\mathcal{M}_{2d}$ itself, with the blown up origin $\lambda = 0$ (the blowup does not affect the space of holomorphic functions, since the origin has very high codimension).

The ghost number is precisely n, the degree of the bundle. The character $Z_D(t,\sigma)$ can be written as:

$$Z_D(t,\vec{\sigma}) = \sum_{n=0}^{\infty} t^n \sum_i (-1)^i \mathrm{Tr}_{\mathbf{H}^i(\mathcal{X}_d,\mathcal{O}(n))}\left[\mathbf{g}\right]. \tag{3.23}$$

If we are to look at the ghost charge only, then the Riemann-Roch formula gives immediately:

$$Z_D(t,0) = \int_{\mathcal{X}_d} \frac{1}{1 - t\ e^{c_1(\mathcal{L})}} \mathrm{Td}_{\mathcal{X}_d} \tag{3.24}$$

[2] The ghost number of these states would be $(0,1,1,2,2,3)$ if one had defined (3.17) to carry ghost-number one as in [10].

where Td is the Todd genus[3]. In terms of the "eigenvalues" x_i of the curvature $\mathcal{R}$ it is given by:

$$\mathrm{Td}_{\mathcal{X}_d} = \prod_i \frac{x_i}{1 - e^{-x_i}} = 1 + \tfrac{1}{2} c_1(\mathcal{X}_d) + \ldots .$$

The full character is given by the equivariant version of the Riemann-Roch formula.

Now let us utilize the $Spin(D)$ group action on $\mathcal{X}_d = \mathbf{P}(\mathcal{M}_{2d})$. Consider the space $\mathbf{H}^0\left(\mathbf{P}(\mathcal{M}_{2d}), \mathcal{O}(n)\right)$. We can view it as a Hilbert space of a quantum mechanical problem, where the phase space is $\mathcal{X}$ and the symplectic form is $nc_1(\mathcal{L})$. Then the trace $\mathrm{Tr}\,[\mathbf{g}]$ can be interpreted as the partition function in this quantum mechanical model with the Hamiltonian $\sigma \cdot M$. It has the path integral representation, which can be viewed as a infinite-dimensional generalization of the Duistermaat-Heckmann formula. As such, it is exactly calculable by a fixed point formula [11] which can be stated in the following generality: Suppose the space $\mathcal{M}$ is acted on by the group G. Suppose also that $\mathcal{M}$ is the cone over the base $\mathcal{X}$, $\mathcal{X} = \mathcal{M}/\mathbf{C}^*$ and the G-action on $\mathcal{M}$ commutes with scalings in $\mathbf{C}^*$. Fix the generic element $\mathbf{g} \in G$. Then:

$$Z(t, \mathbf{g}) = \sum_{f \in \mathcal{X}^{\mathbf{g}}} \frac{1}{\mathrm{Det}_{T_f \mathcal{X}}(1 - \mathbf{g})} \frac{1}{1 - t\chi_f(\mathbf{g})} \tag{3.25}$$

where $\mathcal{X}^{\mathbf{g}} \subset \mathcal{X}$ is the set of points in $\mathcal{X}$ which are invariant under the action of the particular element $\mathbf{g} \in G$. The fiber $\mathcal{L}_f$ of the line bundle $\mathcal{L}$ over $f \in \mathcal{X}$ is the one-dimensional representation of the subgroup $\mathbf{T} \subset G$, generated by $\mathbf{g}$: $\mathbf{T} = \overline{\{\mathbf{g}^n | n \in \mathbf{Z}\}}$. For generic element in a compact Lie group G, the subgroup $\mathbf{T}$ is isomorphic to the Cartan torus of G. In this case the one-dimensional representation is characterized by the character $\chi_f(\mathbf{g})$ (homomorphism from $\mathbf{T}$ to $\mathbf{C}^*$).

Finally, the tangent space to $\mathcal{X}$ at f is a vector representation of $\mathbf{T}$. The denominator in (3.25) contains the so-called Weyl denominator, the determinant of the action of $1 - \mathbf{g}$ on the tangent space to $\mathcal{X}$ at the fixed point.

In the case $D = 11$ the formula (3.25) cannot be applied directly, since the space $\mathcal{M}_{11}$ is a cone over singular variety. However, by further blowups it can be made smooth and the formula like (3.25) can be applied. We discuss this in detail later on.

[3] It differs from the A-hat genus by the factor $e^{\frac{1}{2}c_1(\mathcal{X}_d)}$. The formula (3.24) can be derived using supersymmetric quantum mechanics on $\mathcal{X}_d$.

3.5. Even dimensional pure spinors

In $D = 2d$ dimensions the space $\mathbf{P}(\mathcal{M}_{2d})$ of projective pure spinors coincides with the space of complex structures on the Euclidean space $\mathbf{R}^{2d}$ compatible with the Euclidean structure. This space can be identified with the coset:

$$\mathbf{P}(\mathcal{M}_{2d}) = SO(2d)/U(d) \tag{3.26}$$

which is a complex manifold of complex dimension $d(d-1)/2$, over which there is a line bundle L, associated with the $U(1)$ bundle $\mathcal{N}_{2d} = SO(2d)/SU(d) \longrightarrow \mathbf{P}(\mathcal{M}_{2d})$. The total space of the line bundle L is the moduli space of Calabi-Yau structures on $\mathbf{R}^{2d}$, i.e. the choices of complex structure and a holomorphic top degree form, all compatible with the Euclidean structure. This space is the same as the space of pure spinors in $2d$ dimension.

If the spin zero fields γ describe the map to $\mathcal{M}_{2d}$ and the spin $(1,0)$ fields β describe the cotangent directions to $\mathcal{M}_{2d}$ (the momenta), then the curved β, γ system has the Virasoro central charge which is given by the real dimension of $\mathcal{M}_{2d}$, i.e.:

$$\boxed{c_{\rm Vir} = 2 + d(d-1)} \tag{3.27}$$

The ghost number anomaly also has a geometric meaning. One can easily show that the space of projective pure spinors has positive first Chern class of the tangent bundle:

$$c_1(\mathcal{T}\mathbf{P}(\mathcal{M}_{2d})) = (2d-2)c_1(L) \tag{3.28}$$

The ghost number anomaly is the first Chern class of the square root of the anticanonical bundle $\mathbf{P}(\mathcal{M}_{2d})$. Its topological origin is the same as the shift $k \to k + N$ in the level k $SU(N)$ WZW model.

For pure spinors in any dimension, the Virasoro central charge $c_{\rm Vir}$ and the ghost-number anomaly $a_{\rm ghost}$ are related to measure factors coming from functional integration [12]. One way to see this is to note that if

$$V = (\lambda)^{n_\lambda}(\theta)^{n_\theta}$$

is the state of maximal ghost-number in the pure spinor BRST cohomology, then n_λ and n_θ are related to $c_{\rm Vir}$ and $a_{\rm ghost}$ by

$$n_\lambda = a_{\rm ghost} + \tfrac{1}{2}c_{\rm Vir}, \quad n_\theta = N - \tfrac{1}{2}c_{\rm Vir}, \tag{3.29}$$

where N is the number of components of an unconstrained D-dimensional spinor. For example, when $D = 10$, $n_\lambda = 3$, $n_\theta = 5$, $a_{\rm ghost} = -8$, $c_{\rm Vir} = 22$, and $N = 16$. And when $D = 11$, $n_\lambda = 7$, $n_\theta = 9$, $a_{\rm ghost} = -16$, $c_{\rm Vir} = 46$, and $N = 32$. The equations of (3.29) can be understood from functional integration methods [13][6] since one needs to insert $\frac{1}{2}c_{\rm Vir}$ picture-lowering operators $Y = (C_\alpha\theta^\alpha)\delta(C_\beta\lambda^\beta)$ to absorb the zero modes of the pure spinor λ^α. So the appropriate non-vanishing inner product for tree amplitudes is $\langle\lambda^{n_\lambda}\theta^{n_\theta}(Y)^{\frac{1}{2}c_{\rm Vir}}\rangle$. Using the formulas of (3.29), this non-vanishing inner product contains N θ's and violates ghost-number by $a_{\rm ghost}$ as expected from functional integration.

3.6. Local formula

The group $Spin(2d)$ acts on $\mathbf{P}(\mathcal{M}_{2d})$. Its maximal torus $\mathbf{T}^d$ acts with 2^{d-1} isolated fixed points λ_ϵ, where

$$\epsilon = (\epsilon_1, \ldots, \epsilon_d) \in \mathbf{Z}_2^{d-1}, \qquad \epsilon_a = \pm 1 \quad \text{for} \quad a = 1, \ldots d, \qquad \prod_a \epsilon_a = 1 \tag{3.30}$$

The contribution of the fixed point λ_ϵ to the character $Z_{2d}(t, \sigma)$, as in (3.25), is given by:

$$\begin{aligned} Z_\epsilon \quad &= \quad \frac{1}{1 - ts_\epsilon^{-1}} \prod_{1\leq a<b\leq d} \frac{1}{1 - e^{i(\epsilon_a\sigma_a + \epsilon_b\sigma_b)}} \\ & \quad\quad s_{\vec{\epsilon}} = \prod_a e^{\frac{i}{2}\epsilon_a\sigma_a} \end{aligned} \tag{3.31}$$

where the factors $(1 - e^{i(\epsilon_a\sigma_a+\epsilon_b\sigma_b)})^{-1}$ come from the $d(d-1)/2$ $U(d)$ variables u_{ab} which describe the tangent space near the fixed point. For the case $d = 5$, one can perform the sum over the 16 fixed points and obtain the formula (3.14) for $Z_{10}(t, \sigma)$.

3.7. Non-zero momentum states

In the study of the superparticle in ten dimensions, one deals with the Q operator acting on the space $\mathbb{H}_{X,\theta,\lambda}$ of functions $\Psi(X, \theta, \lambda)$, where X is the vector of $Spin(10)$, $\theta \in \mathbf{S}_-$, is the fermionic spinor, and λ is the pure spinor as before:

$$Q = \lambda\left(\frac{\partial}{\partial\theta} - \theta\gamma^m\frac{\partial}{\partial X^m}\right) \tag{3.32}$$

This operator is Lorentz-invariant. It can be made invariant under the rescaling of λ and θ provided we scale X with the weight 2:

$$\tilde{K} = K + 2X^m\frac{\partial}{\partial X^m}. \tag{3.33}$$

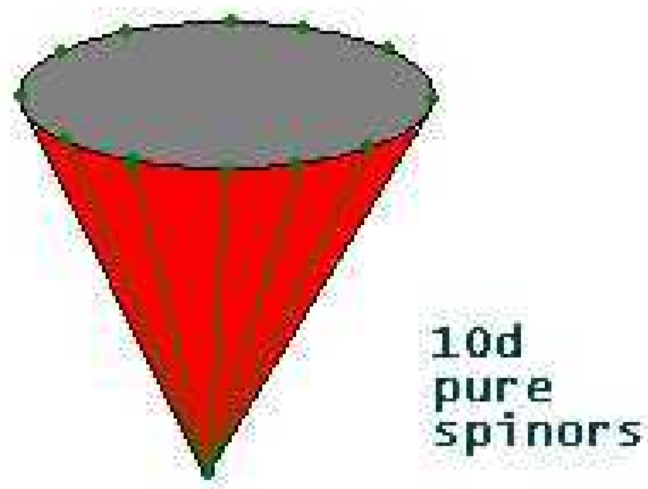

Fig. 1: The space of pure spinors in ten dimensions and the fixed lines entering (3.25)

After multiplying the contribution of (3.13) and (3.14) from the θ and λ variables, one obtains

$$\mathcal{Z}_{10}(t,\sigma) = \mathrm{Tr}_{\mathbb{H}_{X,\theta,\lambda}}(-1)^q t^{\tilde{K}} e^{i\sigma\cdot M} = \frac{P_{10}(t,\sigma)}{\prod_{a=1}^{5}(1-t^2 e^{i\sigma_a})(1-t^2 e^{-i\sigma_a})}. \tag{3.34}$$

We see that the degree 2 term present in Z_{10} is killed in $\mathcal{Z}_{10}$, since the zero momentum state $\lambda\gamma\theta$ is Q-exact once X-dependence is allowed. Note that together $\tilde{K}$ and $M_{2a-1,2a}$, $a = 1,\ldots,5$ form the maximal torus of the conformal group $SO(1,11)$ in ten (Euclidean) dimensions, the symmetry of the classical super-Yang-Mills theory. One can arrange the expansion of $\mathcal{Z}_{10}$ in the characters of $SO(1,11)$ but we shall not need it.

4. Covariant Computation of $D = 10$ Central Charges

In this section we shall calculate various central charges of the worldsheet conformal algebra using the expression (3.14) for the character.

4.1. Generators and relations

Let us forget about the $Spin(10)$ spin content, i.e. let us only look at the dimensions. In other words, set $\vec{\sigma} = 0$. We get:

$$z_{10}(t) = Z_{10}(t,0) = \frac{(1+t)(1+4t+t^2)}{(1-t)^{11}} \tag{4.1}$$

Note the relation:

$$z_{10}(t) = (-1)^{11} t^{-8} z_{10}\left(1/t\right) \tag{4.2}$$

This property of the character is directly related to the ghost number anomaly. We shall return to it later.

Now suppose we have N_n generators/relations at the ghost number n (this number is positive for generators and negative for relations). In other words, suppose that we have the sequence of free fields:

$$\lambda^\alpha, s^m, \sigma_\alpha, \ldots \tag{4.3}$$

which represent the sigma model with the target space $\mathcal{M}$. These fields are related by:

$$s^m = \lambda\gamma^m\lambda, \quad \sigma_\alpha = s_m \left(\gamma^m\lambda\right)_\alpha, \ldots \tag{4.4}$$

and have the ghost numbers 1, 2, 3, ... The fields at ghost number n will have $|N_n|$ components, and will be bosons for $N_n > 0$ and fermions for $N_n < 0$. Then:

$$z_{10}(t) = \prod_{n=1}^{\infty} (1 - t^n)^{-N_n} \tag{4.5}$$

The multiplicities N_n contain the information about the Virasoro central charge, as well as the ghost current algebra:

$$\boxed{\begin{aligned} &\tfrac{1}{2} c_{\mathrm{Vir}} = \sum_n N_n \\ &a_{\mathrm{ghost}} = -\sum_n n N_n \\ &c_{\mathrm{ghost}} = -\sum_n n^2 N_n \end{aligned}} \tag{4.6}$$

which follows from the consideration of the Lagrangian:

$$\mathcal{L} = \int w_\alpha \bar{\partial}\lambda^\alpha - \pi_m \bar{\partial} s^m + p^\alpha \bar{\partial}\sigma_\alpha + \ldots \tag{4.7}$$

We can easily deduce from (4.1)(4.5):

$$
\begin{aligned}
&N_1 = 16\\
&N_2 = -10\\
&N_3 = 16\\
&N_4 = -45\\
&N_5 = 144\\
&N_6 = -456\\
&\dots\\
&N_n \sim (-1)^{n-1}\frac{1}{n}(2+\sqrt{3})^n, \qquad n \to \infty
\end{aligned}
\tag{4.8}
$$

and also the exact relation:

$$
(-1)^{n-1}\left((2+\sqrt{3})^n + (2-\sqrt{3})^n\right) = \sum_{d|n} d\tilde{N}_d \tag{4.9}
$$

where $N_d = \tilde{N}_d$ for $d > 2$, $N_1 = 12 + \tilde{N}_1$, $N_2 = -1 + \tilde{N}_2$.

4.2. Extracting N_n

There is, in fact, an inversion formula for (4.9). It utilizes the so-called Möbious function, $\mu(n)$, which is defined as follows: $\mu(n) = 0$ if n is not integer, or if it is an integer and has one or more repeated prime factors, $\mu(1) = 1$ and

$$
\mu(p_1 \dots p_k) = (-1)^k \tag{4.10}
$$

for distinct primes $p_1, \dots, p_k$. Then (4.9) implies:

$$
d\tilde{N}_d = \sum_k (-1)^{k-1}\mu\left(\frac{d}{k}\right)\left((2+\sqrt{3})^k + (2-\sqrt{3})^k\right) \tag{4.11}
$$

From (4.11) we can get the list of N_n's. Here are the first 30:

$$
\begin{aligned}
&\{N_n\} = \\
&\{16, -10, 16, -45, 144, -456, 1440, -4680, 15600, -52488, \\
&177840, -608160, 2095920, -7264080, \\
&25300032, -88517520, 310927680, \\
&-1095939000, 3874804560, -13737892896, \\
&48829153920, -173949661080, \\
&620963048160, -2220904271040, 7956987570576, -28553733633240, \\
&102617166646800, -369294887482560, \\
&1330702217420400, -4800706688284704\}
\end{aligned}
\tag{4.12}
$$

4.3. Moments of N_n's

The Mellin transform

$$\sum_{n=1}^{\infty} \mu(n) n^{-s} = \frac{1}{\zeta(s)} \tag{4.13}$$

of the Möbius function, which is trivially calculable from the Euler formula

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \tag{4.14}$$

allows to evaluate the regularized moments of the multiplicities N_n:

$$f(s) = \sum_n n^{s+1} N_n = 12 - 2^{s+1} + \tilde{f}(s) \tag{4.15}$$

where

$$
\begin{aligned}
\tilde{f}(s) = \sum_n n^{s+1} \tilde{N}_n &= \sum_{l,k} (-)^{k-1} \mu(l)(kl)^s \left((2+\sqrt{3})^k + (2-\sqrt{3})^k\right) \\
&= \frac{1}{\zeta(-s)} \sum_{k=1}^{\infty} (-)^{k-1} k^s \left((2+\sqrt{3})^k + (2-\sqrt{3})^k\right) \\
&\overset{regularization}{=} \frac{4(2\pi)^s \cos\left(\frac{\pi s}{2}\right)}{\zeta(s+1)} \int_0^{\infty} \frac{dt}{t} t^{-s} \frac{2e^t + 1}{e^{2t} + 4e^t + 1}
\end{aligned}
\tag{4.16}
$$

The last line allows analytic continuation of $\tilde{f}(s)$. The integral is convergent for $s < 0$. We are interested in the cases $s = -1, 0, 1$. We have, from (4.6):

$$
\begin{aligned}
&\tfrac{1}{2}c_{\text{Vir}} = f(-1) = 11 \\
&a_{\text{ghost}} = -f(0) = -12 + 2 - \frac{1}{\zeta(0)}\left(\frac{2+\sqrt{3}}{3+\sqrt{3}} + \frac{2-\sqrt{3}}{3-\sqrt{3}}\right) = -8 \\
&c_{\text{ghost}} = -f(1) = -12 + 4 - \frac{1}{\zeta(-1)}\left(\frac{2+\sqrt{3}}{(3+\sqrt{3})^2} + \frac{2-\sqrt{3}}{(3-\sqrt{3})^2}\right) = -4
\end{aligned}
\tag{4.17}
$$

4.4. Ghost number anomaly and central charge without knowing N_n

Consider the general expression of the form we analyzed in the previous section:

$$
\prod_n (1-t^n)^{-N_n} = \frac{P(t)}{Q(t)} \tag{4.18}
$$

where P and Q are some polynomials. We have:

$$
\sum_n N_n \log(1-e^{nx}) = -\log\frac{P(e^x)}{Q(e^x)} \tag{4.19}
$$

Since

$$
\log(1-e^x) = \log(-x) + \frac{x}{2} + \sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g} \tag{4.20}
$$

where B_k are Bernoulli numbers, we have:

$$
\begin{aligned}
\log(-x)\sum_n N_n + \sum_n \log(n)N_n + \frac{x}{2}\sum_n nN_n + \\
\sum_{g=1}^{\infty}\frac{B_{2g}}{2g(2g)!}x^{2g}\sum_n n^{2g}N_n = -\log\frac{P(x)}{Q(x)}
\end{aligned}
\tag{4.21}
$$

Expanding the right hand side of (4.21) we get all the even moments of the multiplicity function N_n. To get the odd moments besides $\sum_n N_n$ we ought to use the more sophisticated machinery of the previous section.

In ten dimensions we have:

$$
\begin{aligned}
&P_{10}(t) = (1+t)(1+4t+t^2), \qquad Q_{10}(t) = (1-t)^{11} \\
&\log P_{10}(e^x)/Q_{10}(e^x) = -11\log(-x) + \log(12) - 4x - \frac{x^2}{6} + \ldots
\end{aligned}
\tag{4.22}
$$

and, therefore:

$$
\begin{gathered}
\boxed{\tfrac{1}{2}c_{\mathrm{Vir}} = \sum_n N_n = 11} \\
\sum_n \log(n) N_n = -\log(12) \\
\boxed{a_{\mathrm{ghost}} = -\sum_n n N_n = -8} \\
\boxed{c_{\mathrm{ghost}} = -\sum_n n^2 N_n = -4}
\end{gathered}
\tag{4.23}
$$

(recall that $B_2 = \frac{1}{6}$). Even though it is not obviously interesting at the current stage, we also list the next few moments:

$$
\begin{aligned}
\sum_n n^4 N_n &= 4 \\
\sum_n n^6 N_n &= 4 \\
\sum_n n^8 N_n &= -\frac{68}{3}
\end{aligned}
\tag{4.24}
$$

4.5. Lorentz current central charge

Let us go back to the general character (3.14) and let us specialize:

$$
\sigma_2 = \ldots = \sigma_5 = 0, \qquad e^{\frac{i\sigma_1}{2}} = q \tag{4.25}
$$

Then

$$
\begin{aligned}
&z_{10}(t, q) \equiv Z_{10}(t, \sigma_1, 0, 0, 0, 0) = \\
&= \frac{(1-t^2)(1 + t(1+t^2)(q+q^{-1}) - 6t^2 + t^4)}{(1-tq)^7(1-tq^{-1})^7} \\
&= \prod_{n\geq 1, m\in \mathbf{Z}} (1 - t^n q^m)^{N_{n,m}}
\end{aligned}
\tag{4.26}
$$

We are interested in

$$
c_{\mathrm{Lorentz}} = -\sum_{m,n} \frac{m^2}{4} N_{n,m} \tag{4.27}
$$

We start with:

$$\sum_{n,m} N_{n,m} \log(1 - t^n q^m) = \log\ z_{10}(t,q) \tag{4.28}$$

from which we derive:

$$\begin{aligned}\sum_{n,m} \frac{m^2}{4} N_{n,m} \frac{1}{(t^{\frac{n}{2}} - t^{-\frac{n}{2}})^2} =& - \partial_x^2|_{x=0} \log\ z_{10}(t, e^{\frac{x}{2}}) \\ & - \frac{2t(2+7t+2t^2)}{(1-t)^2(1+4t+t^2)} = \\ & - \frac{11}{3(\log(t))^2} + \frac{1}{4} - \frac{131(\log(t))^2}{2160} + \dots\end{aligned} \tag{4.29}$$

and:

$$\begin{gathered}\sum_{n,m} \frac{m^2}{4n^2} N_{n,m} = -\frac{11}{3} \\ \boxed{c_{\text{Lorentz}} = -\sum_{n,m} \frac{1}{4} m^2 N_{n,m} = -3} \\ \sum_{n,m} \frac{1}{4} m^2 n^2 N_{n,m} = \frac{13}{9}\end{gathered} \tag{4.30}$$

4.6. The ghost number anomaly revisited

Let us look at the general formula for Z_d:

$$Z_{2d}(t,\vec{\sigma}) = \sum_{\vec{\epsilon},\prod_a \epsilon_a = 1} \frac{1}{1-ts_{\vec{\epsilon}}^{-1}} \prod_{1\le a<b\le d} \frac{1}{1-e^{i(\epsilon_a\sigma_a+\epsilon_b\sigma_b)}} \tag{4.31}$$

and let us define the conjugate character by:

$$Z^*_{2d}(t,\vec{\sigma}) = \sum_{\vec{\epsilon},\prod_a \epsilon_a = -1} \frac{1}{1-ts_{\vec{\epsilon}}} \prod_{1\le a<b\le d} \frac{1}{1-e^{-i(\epsilon_a\sigma_a+\epsilon_b\sigma_b)}}. \tag{4.32}$$

For odd d the conjugate character counts the holomorphic functions on the space of pure spinors of the opposite chirality. For example, in ten dimensions this exchanges $\mathbf{S}_+$ with $\mathbf{S}_-$ in all our formulae above. For even d, $Z^*_{2d} = Z_{2d}$. Clearly, $Z^*_{2d}(t,0) = Z_{2d}(t,0)$ for any d.

On the other hand, an elementary calculation shows:

$$Z^*_{2d}(t,\sigma) = (-1)^{\Delta}\left(t^{-2(d-1)}Z_{2d}(1/t,\sigma) + \sum_{l=0}^{2(d-2)} t^{l+3-2d}\tilde{z}_{-l-1}(\sigma)\right) \tag{4.33}$$

where $\Delta = 1 + d(d-1)/2$ is the complex dimension of $\mathcal{M}_d$, and $\tilde{z}_{-l-1}$ is the coefficient at $-t^{-l-1}$ in the expansion of $Z_{2d}(t,\vec{\sigma})$ near $t=\infty$. In all our examples $\tilde{z}_{-l-1} = 0$ for $0 \le l \le 2d-4$. On the other hand, the inspection of the Riemann-Roch formula (3.24) gives:

$$Z_{2d}(t,0) = (-1)^{\Delta}\left(t^{-\delta}Z_{2d}(1/t,0) + \sum_{l=0}^{2(d-2)} t^{l+3-2d}\tilde{z}_{-l-1}(0)\right) \tag{4.34}$$

for the manifold $\mathcal{X}$ such that $c_1(\mathcal{X}) = \delta c_1(L)$. Thus, $\delta = 2d-2$.

5. Eleven-dimensional Pure Spinors

In eleven dimensions, $D=11$, there exists a notion of pure spinors which seems suitable for quantization of superparticle and, perhaps, supermembrane as well[5]. The space $\mathcal{M}_{11}$ of eleven dimensional pure spinors is the space of spinors λ, obeying

$$\lambda^{\alpha}\gamma_{\alpha\beta}{}^{I}\lambda^{\beta} = 0, \qquad I = 0,\ldots,10, \qquad \alpha = 1,\ldots,32. \tag{5.1}$$

If we decompose $\lambda = (\lambda_L, \lambda_R)$ the eleven dimensional spinor as the sum of the left- and right-chirality ten dimensional spinors, the equation (5.1) reads as:

$$\begin{aligned} &\lambda_L \gamma^\mu \lambda_L + \lambda_R \gamma^\mu \lambda_R = 0, \qquad \mu = 1, \dots 10 \\ &\lambda_L \lambda_R = 0 \end{aligned} \tag{5.2}$$

The space of eleven dimensional projective pure spinors $\mathbf{P}(\mathcal{M}_{11})$ is the quotient of the space of solutions to (5.2) with the point $\lambda = 0$ deleted, by the action of the group $\mathbf{C}^*$ of scalings: $\lambda \mapsto s\lambda$, $s \neq 0$. It has (complex) dimension 22.

5.1. Global formula

In this case, the formula for the character (which has the same arguments as in ten dimensional case, since $Spin(11)$ also has rank five) is given by:

$$\boxed{Z_{11}(t, \sigma) \quad = \quad \frac{P_{11}(t,\sigma)}{Q_{11}(t,\sigma)}}$$

$$\boxed{\begin{aligned} P_{11}(t,\sigma) \quad = \quad & 1 - t^2 V + t^4 (V + \Lambda^2 V) \\ & - t^5 S - t^6 (\Lambda^3 V + \mathrm{Sym}^2 V) + t^7 V S \\ & - t^9 V S + t^{10} (\Lambda^3 V + \mathrm{Sym}^2 V) + t^{11} S \\ & - t^{12} (V + \Lambda^2 V) + t^{14} V - t^{16} \end{aligned}} \tag{5.3}$$

$$\boxed{Q_{11}(t,\sigma) \quad = \quad \prod_{\pm\pm\pm\pm\pm} \left(1 - t e^{\frac{i}{2}(\pm\sigma_1 \pm\sigma_2 \pm\sigma_3 \pm\sigma_4 \pm\sigma_5)}\right)}$$

where:

$$\begin{aligned} V = 1 + 2\sum_{a=1}^{5} \cos(\sigma_a) = \sum_{I=0}^{10} e^{y_I}\,,\ S = \prod_{a=1}^{5} \left(e^{-\frac{i\sigma_a}{2}} + e^{\frac{i\sigma_a}{2}}\right), \\ \mathrm{Sym}^2 V = \sum_{I \leq J} e^{y_I + y_J}\,,\ \Lambda^2 V = \sum_{I<J} e^{y_I + y_J}\,,\ \Lambda^3 V = \sum_{I<J<K} e^{y_I + y_J + y_K} \end{aligned} \tag{5.4}$$

are the characters of the vector, spinor[4], symmetric tensor, antisymmetric rank two tensor, and antisymmetric rank three tensor representations of $Spin(11)$ respectively.

As in the $D=10$ case discussed earlier, one can derive the above formula using either the reducibility method or the fixed-point method. Although the reducibility method is considerably more complicated than in $D=10$, its relation to the physical states of the $D=11$ superparticle allows one to derive the appropriate reducibility conditions. On the other hand, the fixed-point method is more straightforward. However, its relation to the physical states is somewhat indirect.

5.2. Reducibility method

To derive the formula of (5.3) using the reducibility method, note that $Z_{11}(t,\sigma)$ can be written as $P_{11}(t,\sigma)/Q_{11}(t,\sigma)$ where $(Q_{11}(t,\sigma))^{-1}$ is the partition function for an unconstrained 32-component spinor. It will now be shown that the numerator $P(t,\sigma)$ of (5.3) comes from the constraints implied by the $D=11$ pure spinor condition $\lambda\gamma^I\lambda=0$ where $I=0$ to 10.

As in the $D=10$ case, the $D=11$ pure spinor constraint and its reducibility conditions are closely related to the zero momentum states of the $D=11$ superparticle. As discussed in the appendix of [5], the reducibility conditions for the $D=11$ pure spinor constraint $\lambda\gamma^I\lambda=0$ can be described by the nilpotent operator

$$\widetilde{Q}=\widetilde{\lambda}\Gamma^I\widetilde{\lambda}b_{(-1)I}+c^I_{(1)}(\widetilde{\lambda}\Gamma_{IJ}\widetilde{\lambda}u^J_{(-2)}+\widetilde{\lambda}\Gamma^J\widetilde{\lambda}u_{(-2)[IJ]}) \tag{5.5}$$

$$+v^I_{(2)}((\widetilde{\lambda}\Gamma_I)_\alpha b^\alpha_{(-2)}+\widetilde{\lambda}\Gamma^J\widetilde{\lambda}b_{(-3)(IJ)})$$

$$+v^{[IJ]}_{(2)}(\tfrac{1}{2}(\widetilde{\lambda}\Gamma_{IJ})_\alpha b^\alpha_{(-2)}+\eta_{JK}\widetilde{\lambda}\Gamma^{KL}\widetilde{\lambda}b_{(-3)(IL)}+\widetilde{\lambda}\Gamma^K\widetilde{\lambda}b_{(-3)[IJK]})$$

$$+c^\alpha_{(2)}(-\widetilde{\lambda}\Gamma^I\widetilde{\lambda}u_{(-3)I\alpha}+\tfrac{1}{2}(\widetilde{\lambda}\Gamma^{IJ})_\alpha(\widetilde{\lambda}\Gamma_I)^\beta u_{(-3)J\beta}))$$

$$+\tfrac{1}{2}c^{(JK)}_{(3)}(\widetilde{\lambda}\Gamma_J)^\alpha u_{(-3)K\alpha}+\frac{1}{4}c^{[JKL]}_{(3)}(\widetilde{\lambda}\Gamma_{KL})^\alpha u_{(-3)J\alpha}$$

$$+v^{I\alpha}_{(3)}b^{J\beta}_{(-4)}M_{I\alpha\ J\beta\ \gamma\delta}\widetilde{\lambda}^\gamma\widetilde{\lambda}^\delta$$

$$+\frac{1}{4}u^{[JKL]}_{(-4)}(\widetilde{\lambda}\Gamma_{KL})^\alpha c_{(4)J\alpha}+\tfrac{1}{2}u^{(JK)}_{(-4)}(\widetilde{\lambda}\Gamma_J)^\alpha c_{(4)K\alpha}$$

$$+u^\alpha_{(-5)}(-\widetilde{\lambda}\Gamma^I\widetilde{\lambda}c_{(4)I\alpha}+\tfrac{1}{2}(\widetilde{\lambda}\Gamma^{IJ})_\alpha(\widetilde{\lambda}\Gamma_I)^\beta c_{(4)J\beta})$$

[4] Note that $S=\sum_{\pm\pm\pm\pm\pm}e^{\frac{i}{2}(\pm\sigma_1\pm\sigma_2\pm\sigma_3\pm\sigma_4\pm\sigma_5)}$

$$+b^{[IJ]}_{(-5)}(\tfrac{1}{2}(\widetilde{\lambda}\Gamma_{IJ})_\alpha v^\alpha_{(5)} + \eta_{JK}\widetilde{\lambda}\Gamma^{KL}\widetilde{\lambda}v_{(4)(IL)} + \widetilde{\lambda}\Gamma^K\widetilde{\lambda}v_{(4)[IJK]})$$

$$+b^I_{(-5)}((\widetilde{\lambda}\Gamma_I)_\alpha v^\alpha_{(5)} + \widetilde{\lambda}\Gamma^J\widetilde{\lambda}v_{(4)(IJ)})$$

$$+u^I_{(-6)}(\widetilde{\lambda}\Gamma_{IJ}\widetilde{\lambda}c^J_{(5)} + \widetilde{\lambda}\Gamma^J\widetilde{\lambda}c_{(5)[IJ]}) + b_{(-7)}\widetilde{\lambda}\Gamma^I\widetilde{\lambda}v_{(6)I},$$

where $M_{I\alpha\ J\beta\ \gamma\delta}$ are fixed coefficients, and $\widetilde{\lambda}^\alpha$ is an unconstrained spinor.

In order that $\widetilde{Q}$ has ghost-number zero, the variables

$$[1, c^I_{(1)}, v^I_{(2)}, v^{[IJ]}_{(2)}, c^\alpha_{(2)}, c^{(IJ)}_{(3)}, c^{[IJK]}_{(3)}, v^{I\alpha}_{(3)}, c^{I\alpha}_{(4)}, v^{[IJK]}_{(4)}, v^{(IJ)}_{(4)}, v^\alpha_{(5)}, c^{[IJ]}_{(5)}, c^I_{(5)}, v^I_{(6)}, c_{(7)}], \quad (5.6)$$

carry respectively ghost-number[5]

$$[0, 2, 4, 4, 5, 6, 6, 7, 9, 10, 10, 11, 12, 12, 14, 16]. \quad (5.7)$$

As in the $D = 10$ case, one can check that the coupling of the variables in (5.5) describes the reducibility conditions satisfied by $\widetilde{\lambda}\gamma^I\widetilde{\lambda} = 0$.

By comparing (5.5) with (5.3), one can easily check that each term in $P(t, \sigma)$ corresponds to a reducibility condition. For example, the term $-t^2 V$ corresponds to the original pure spinor constraint, the term $t^4(V + \Lambda^2 V)$ corresponds to the reducibility conditions described by the second term in (5.5), the terms $-t^5 S - t^6(\Lambda^3 V + \mathrm{Sym}^2 V)$ correspond to the reducibility conditions described by the second and third lines in (5.5), etc. So, as claimed, $Z_{11}(t, \sigma) = P_{11}(t, \sigma)/Q_{11}(t, \sigma)$ correctly computes the character formula.

As discussed in [5], the operator of (5.5) is related to the zero momentum spectrum of the $D = 11$ superparticle which describes linearized supergravity. It was shown that the supergravity ghosts, fields, antifields and antighosts are described by the states of (5.6). As in $D = 10$, the fact that these states describe the zero momentum spectrum can also be seen by computing the cohomology of (3.2). Since the contribution from the partition function for θ cancels the denominator of (5.3), the cohomology of (3.2) is described by the numerator $P_{11}(t, \sigma)$ of (5.3).

---

[5] In the appendix of [5], $\widetilde{Q}$ was defined to carry ghost-number one, which implied that the ghost numbers coincide with the subscript on the variables. In this paper, $\widetilde{Q}$ will be defined to carry ghost-number zero with respect to K of (3.3) so that the t dependence counts the ghost number.

5.3. Fixed point method

The global formula (5.3) can also be computed as the sum of 2^5 local contributions, which correspond to the fixed points of the action of the element $\tilde{g} = (\mathbf{g}, 1)$ on $\mathbf{P}(\mathcal{M}_{11})$, where $\mathbf{g}$ is as in (3.8). Unlike the $D = 10$ case where the local contribution is straightforward to calculate, the $D = 11$ story is more involved. The difference is the singular nature of the space of projective pure spinors in $D = 11$. Let us examine the neighbourhood of a fixed point of the element $\tilde{g}$ action on the space $\mathbf{P}(\mathcal{M}_{11})$.

First of all, the element $\tilde{g}$ of $Spin(11)$ fixes the point f on $\mathbf{P}(\mathcal{M}_{11})$, $\tilde{g} \cdot f = f$, iff it preserves a line $s \cdot \lambda$, $s \neq 0$, on $\mathcal{M}_{11}$. In other words, there should be a pure spinor λ_f, which is an eigenvector of $\tilde{g}$. The fixed points $f \in \mathbf{P}(\mathcal{M}_{11})$ are in one-to-one correspondence with such spinors.

The eigenvectors of $\tilde{g}$ in the space of all spinors are in one-to-one correspondence with the fivetuples of plus and minus signs $\pm\pm\pm\pm\pm$. The corresponding eigenvalue is

$$e^{\frac{i}{2}(\pm\sigma_1 \pm\sigma_2 \pm\sigma_3 \pm\sigma_4 \pm\sigma_5)}$$

We can choose any of these. Thus, we get 2^5 fixed points λ_ϵ, labeled by the vectors $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \in \mathbf{Z}_2^5$, where for each $a = 1, \ldots, 5$, $\epsilon_a = \pm 1$. The group $\mathbf{Z}_2^5 = W_{B_5}/W_{A_4}$ is the quotient of the Weyl group of $Spin(11)$ by that of $SU(5)$, the subgroup of $Spin(11)$ preserving the spinor λ_f . Introduce the notations:

$$\begin{aligned} w_a &= e^{i\epsilon_a \sigma_a} \\ s &= \left(\prod_{a=1}^{5} w_a\right)^{\frac{1}{2}} \\ g_a &= s w_a^{-1} \\ \Sigma_\pm &= \sum_{a=1}^{5} w_a^{\pm 1} \end{aligned} \tag{5.8}$$

Then the contribution of the fixed point ϵ is given by:

$$Z_{11;\epsilon} = \frac{1 - s^2\Sigma_+ + s^4\Sigma_- - s^6}{1 - ts^{-1}} \prod_{m=1}^{3} \prod_{a_1<a_2<\ldots<a_m} \frac{1}{(1 - w_{a_1} w_{a_2} \ldots w_{a_m})} \tag{5.9}$$

where the various factors appearing in $Z_{11;\epsilon}$ will be derived below. Using the formula of (5.9), one can derive the formula of (5.3) by summing over the 32 fixed points as

$$Z_{11}(t, \sigma) = \sum_{\epsilon \in \mathbf{Z}_2^5} Z_{11;\epsilon} \ .$$

<u>Derivation of the local contribution :</u>

Because of the numerator, the factor

$$(1 - s^2\Sigma_+ + s^4\Sigma_- - s^6) \prod_{m=1}^{3} \prod_{a_1<a_2<\ldots<a_m} \frac{1}{1 - e^{i(\epsilon_{a_1}\sigma_{a_1}+\ldots+\epsilon_{a_m}\sigma_{a_m})}} \tag{5.10}$$

in (5.9) differs in form from the typical fixed point contribution, announced in (3.25). This is because the point λ_ϵ is singular, and one needs a more careful analysis of its neighbourhood. We now present this analysis.

The factors in the denominator in (5.10) correspond to the parameterization

$$u_a \sim e^{i\epsilon_a\sigma_a}, \quad u_{ab} \sim e^{i(\epsilon_a\sigma_a+\epsilon_b\sigma_b)}, \quad u_{abc} \sim e^{i(\epsilon_a\sigma_a+\epsilon_b\sigma_b+\epsilon_c\sigma_c)}, \tag{5.11}$$

of the 22-dimensional tangent space near the fixed point. Since (u_a, u_{ab}, u_{abc}) describe 25 parameters, there will be three constraints on these parameters. Let us discuss this parameterisation in more detail. For each split $\mathbf{R}^{11} = \mathbf{R} \oplus \mathbf{C}^5$ the 32-dimensional space of all spinors can be identified with the space of all p-forms, $p = 0, 1, 2, 3, 4, 5$ on $\mathbf{C}^5$, modulo the factor $\left(\Lambda^5\mathbf{C}^5\right)^{-\frac{1}{2}}$:

$$\mathbf{S} = \bigoplus_{p=0}^{5} \Lambda^p\mathbf{C}^5 \otimes \left(\Lambda^5\mathbf{C}^5\right)^{-\frac{1}{2}}. \tag{5.12}$$

In other words, any spinor can be expressed as $\sum_{p=0}^{5} a_{j_1\cdots j_p}\lambda^{j_1\cdots j_p}$ where $a_{j_1\cdots j_p}$ are coefficients and $\lambda^{j_1\cdots j_p}$ has eigenvalue $e^{i(\sigma_{j_1}+\ldots+\sigma_{j_p})}\left(e^{i(\sigma_1+\ldots+\sigma_5)}\right)^{-\frac{1}{2}}$. It is convenient to choose the eigenvector λ_ϵ to be in the component $\Lambda^0\mathbf{C}^5 \otimes \left(\Lambda^5\mathbf{C}^5\right)^{-\frac{1}{2}}$, i.e. $a_{j_1\cdots j_p} = 0$ for $p > 0$. Now let us consider the infinitesimal variations of this spinor:

$$\lambda = \lambda_f\left(1 + v^{(1)} + v^{(2)} + v^{(3)} + v^{(4)} + v^{(5)}\right) = \lambda_f\left(1 \oplus v_a \oplus v_{ab} \oplus v_{abc} \oplus v_{abcd} \oplus v_{abcde}\right)$$

where the m-form $v_{a_1\ldots a_m}$ has the eigenvalue $e^{i(\epsilon_{a_1}\sigma_{a_1}+\ldots+\epsilon_{a_m}\sigma_{a_m})}$ under the $\tilde{g}$ action.

The pure spinor constraint (5.1) reads in this parameterization, symbolically:

$$\begin{aligned} &v^{(4)} + v^{(2)}v^{(2)} + v^{(1)}v^{(3)} = 0 \\ &v^{(5)} + v^{(1)}v^{(4)} + v^{(2)}v^{(3)} = 0 \\ &v^{(1)}v^{(5)} + v^{(2)}v^{(4)} + v^{(3)}v^{(3)} = 0 \end{aligned} \tag{5.13}$$

We can solve the first two equations by:

$$\begin{aligned}
v^{(p)} &= \varepsilon u^{(p)} \quad \text{for} \quad p = 1,2,3 \\
v^{(4)} &= -\varepsilon^2 \left(u^{(2)}u^{(2)} + u^{(1)}u^{(3)} \right) \\
v^{(5)} &= -\varepsilon^2 u^{(2)}u^{(3)} + \varepsilon^3 u^{(1)} \left(u^{(2)}u^{(2)} + u^{(1)}u^{(3)} \right)
\end{aligned} \tag{5.14}$$

where ε is any complex number. Now the third equation in (5.13) reads (again, symbolically):

$$u^{(3)}u^{(3)} + \varepsilon \left[-u^{(2)}u^{(2)}u^{(2)} + u^{(1)}u^{(2)}u^{(3)} \right] + \varepsilon^2 \left[u^{(1)}u^{(1)} \left(u^{(2)}u^{(2)} + u^{(1)}u^{(3)} \right) \right] = 0 \tag{5.15}$$

For small ε, these equations imply that the infinitesimal deformations of the pure spinor λ_ϵ are parameterized by the arbitrary one-form $u^{(1)} = (u_a)$, the arbitrary two-form $u^{(2)} = (u_{ab})$ and the constrained three-form $u^{(3)} = (u_{abc})$ which obeys

$$u^{(3)}u^{(3)} = 0 \Leftrightarrow \epsilon^{abcde} u_{abc} u_{def} = 0 \tag{5.16}$$

We now have to calculate the character of the $\tilde{g}$ action on the space of holomorphic functions of $u^{(1)}, u^{(2)}, u^{(3)}$ subject to (5.16).

The unconstrained scalar λ_f, the one form $u^{(1)}$ and the two-form $u^{(2)}$ contribute the factors $\chi^{(0)}$, $\chi^{(1)}$ and $\chi^{(2)}$, respectively:

$$\begin{aligned}
\chi^{(0)} &= \frac{1}{1 - t s_\epsilon^{-1}} \\
\chi^{(1)} &= \prod_{a=1}^{5} \frac{1}{1 - w_a} \\
\chi^{(2)} &= \prod_{1 \le a < b \le 5} \frac{1}{1 - w_a w_b}
\end{aligned} \tag{5.17}$$

where

$$w_a = e^{i\epsilon_a \sigma_a} \, . \tag{5.18}$$

Because of the constraint of (5.16), the contribution from $u^{(3)}$ is more complicated and will be computed below using three alternative methods.

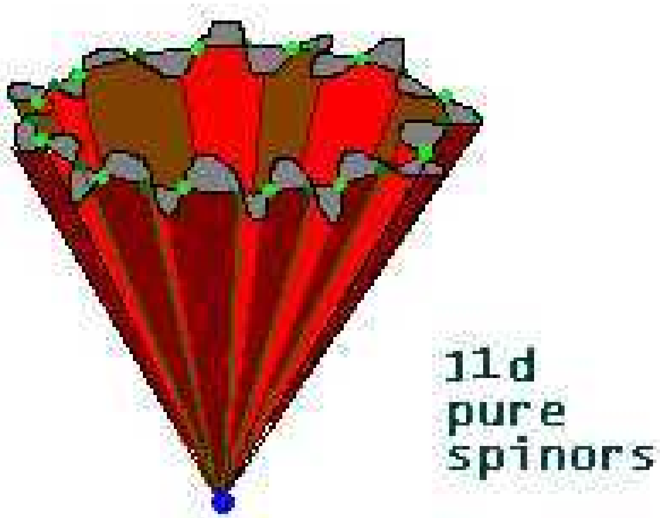

Fig. 2: The space of pure spinors in eleven dimensions and its singularities

<u>$u^{(3)}$ contribution and the Grassmanian $Gr(2,5)$:</u>

The first method for computing the $u^{(3)}$ contribution uses that (5.16) is the Plucker relation, describing the Grassmanian of two-planes in $\mathbf{C}^5$ using the decomposable two-forms. In other words, the equation (5.16) can be solved by:

$$u_{abc} = \epsilon_{abcde} K_1^d K_2^e \tag{5.19}$$

where the 2×5 matrix $K = (K_\alpha^d)$ is defined up to the multiplication by the $SL_2(\mathbf{C})$ matrix which acts on the lower index. This gives, by the way, a simple way to understand the dimension of $\mathcal{M}_{11}$. Indeed, the space of K's is 10 dimensional. Dividing by SL_2 reduces the dimension by 3, and together with the dimensions 1, 5 and 10 coming from λ_f, $u^{(1)}$ and $u^{(2)}$ respectively, we get 23 announced above.

The contribution $\chi^{(3)}$ of $u^{(3)}$ to the character can be therefore expressed as the SL_2 invariant part of the character of the space of functions of the matrix elements K_α^d. In other words, we should calculate

$$\mathrm{Tr}_{\mathrm{Fun}(K)} \left(\tilde{g} \times h \right) \tag{5.20}$$

where $h \in SL_2(\mathbf{C})$, and then average it with respect to h. This is done by integrating over the maximal compact subgroup of SL_2, i.e. $SU(2)$, with respect to the Haar measure on $SU(2)$. Finally, since the character depends only on the conjugacy class of h, the

integral reduces to the integral over the maximal torus of $SU(2)$, i.e. $U(1)$. However, the measure of integration contains an extra factor, the Weyl-Vandermonde determinant. For the details of such computations see [14]. So, take

$$h = \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix}$$

The matrix elements K_1^a are eigenvectors of $\tilde{g} \times h$ with the eigenvalue $g_a e^{\frac{i\alpha}{2}}$ where

$$g_a = e^{-i\epsilon_a \sigma_a} s$$

while K_2^a is the eigenvector with the eigenvalue $g_a e^{-\frac{i\alpha}{2}}$. So (5.20) is equal to:

$$\mathrm{Tr}_{\mathrm{Fun}(K)} \left(\tilde{g} \times h\right) = \prod_{a=1}^{5} \frac{1}{(1 - g_a e^{-\frac{i\alpha}{2}})(1 - g_a e^{\frac{i\alpha}{2}})} \tag{5.21}$$

The averaging over $SU(2)$ is done by the integral:

$$\chi^{(3)} = \mathrm{Tr}_{\mathrm{Fun}(u^{(3)})} \tilde{g} = \frac{1}{\pi} \int_0^{2\pi} \mathrm{d}\alpha \, \sin^2 \left(\frac{\alpha}{2}\right) \, \mathrm{Tr}_{\mathrm{Fun}(K)} \tilde{g} \times h \tag{5.22}$$

which can be evaluated by deforming the contour of integration:

$$\chi^{(3)} = -\frac{1}{4\pi i} \oint \frac{\mathrm{d}z}{z} \left(z - z^{-1}\right)^2 \prod_{a=1}^{5} \frac{1}{(1 - g_a z)(1 - g_a z^{-1})} \tag{5.23}$$

and taking the sum over five residues, at $z = g_a$, $a = 1, \ldots, 5$, to get:

$$\chi^{(3)} = \frac{1 - s^2 \Sigma_+ + s^4 \Sigma_- - s^6}{\prod_{a<b<c} (1 - w_a w_b w_c)}. \tag{5.24}$$

Multiplying the contributions of (5.17) and (5.24), one obtains the desired formula of (5.9).

$u^{(3)}$ contribution from an ultra – local formula :

A second method for computing $\chi^{(3)}$ gives rise to an *ultra-local* formula for $\chi^{(3)}$. Although the five residues of (5.23) do not correspond to any fixed points, the trace $\mathrm{Tr}_{\mathrm{Fun}(u^{(3)})}(\tilde{g})$ has a fixed point expression since $Gr(2,5)$ is a smooth manifold on which $\tilde{g}$ acts with isolated fixed points. On the space of $u^{(3)}$'s this action lifts to the action with fixed projective lines. There are $10 = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ of them. Namely, to each pair $[ab]$, $1 \leq a < b \leq 5$ there corresponds a two-plane in $\mathbf{C}^5$ spanned by the basis vectors $\mathbf{e}_a$ and $\mathbf{e}_b$. Dualizing, the fixed points are in one-to-one correspondence with triples $[abc]$,

$1 \leq a < b < c \leq 5$, which label the three-planes (in the dual $\mathbf{C}^5$) spanned by the coordinate axes $\mathbf{e}^a$, $\mathbf{e}^b$ and $\mathbf{e}^c$. The solution to (5.16), corresponding to $[abc]$ has $u_{abc} \neq 0$ and the rest of the components of $u^{(3)}$ vanishing. The contribution of this fixed line to $\chi^{(3)}$ is:

$$\chi^{(3)}_{abc} = \frac{1}{1 - w_a w_b w_c} \prod_{d \neq a,b,c} \frac{1}{\left(1 - w_a^{-1} w_d\right)\left(1 - w_b^{-1} w_d\right)\left(1 - w_c^{-1} w_d\right)} \tag{5.25}$$

The seven factors in the denominator above come from the following deformations of the $[abc]$ solution to (5.16): the first factor corresponds to the scaling of the u_{abc} component, while the rest comes from the homogeneous components of the form: u_{dbc}/u_{abc}, u_{adc}/u_{abc} and u_{abd}/u_{abc}, respectively. Summing over all the triples $[abc]$ we arrive at the expression (5.24) for $\chi^{(3)}$:

$$\sum_{a<b<c} \chi^{(3)}_{abc} = \chi^{(3)}, \tag{5.26}$$

as can be verified directly, or using another contour integral representation[6]. Finally, combining all the factors we can write the *ultra-local* formula:

$$\begin{aligned} & Z_{11;abc,\epsilon} = \\ & \frac{1}{(1 - t s_\epsilon^{-1})(1 - w_a w_b w_c)} \times \\ & \prod_f \frac{1}{1 - w_f} \times \\ & \prod_{g<h} \frac{1}{1 - w_g w_h} \times \\ & \prod_{d \neq a,b,c} \frac{1}{\left(1 - w_a^{-1} w_d\right)\left(1 - w_b^{-1} w_d\right)\left(1 - w_c^{-1} w_d\right)} \end{aligned} \tag{5.27}$$

so that:

$$Z_{11} = \sum_{\epsilon, a<b<c} Z_{11;abc,\epsilon} \tag{5.28}$$

Note that the triples $a < b < c$ together with ϵ enumerate precisely the cosets $W_{B_5}/W_{A_1} \times W_{A_2}$, where $W_{A_1} \times W_{A_2}$ is the Weyl group of $SU(2) \times SU(3)$ – the stability group of

[6] Since $Gr(2,5) = Gr(3,5)$ there is also a SL_3-invariant formalism, where one writes $u_{abc} = M_a^\alpha M_b^\beta M_c^\gamma \varepsilon_{\alpha\beta\gamma}$, with 5×3 matrix M defined up to the $SL_3(\mathbf{C})$ action on the index α. The corresponding projection onto $SU(3)$ invariant states is accomplished by the contour integral over the two dimensional maximal torus. The residues in this integral are in one-to-one correspondence with the $[abc]$ triples.

$\lambda_f+\lambda_f u^{(3)}$. Also note that the quotient $Spin(11)/SU(2)\times SU(3)$ has dimension $44 = 2\times 22$ which coincides with the dimension of $\mathbf{P}(\mathcal{M}_{11})$. In a sense, $Spin(11)/SU(2)\times SU(3)$ is a local model of $\mathbf{P}(\mathcal{M}_{11})$ near the fixed lines.

$u^{(3)}$ contribution via reducibility constraints :

The third method for computing the $u^{(3)}$ contribution uses the reducibility properties of the constraint (5.16)

$$A_f = \epsilon^{abcde} u_{abc} u_{def} = 0 \ . \tag{5.29}$$

Note that A_f satisfies the identity

$$B^e = \epsilon^{abcde} A_a u_{bcd} = 0, \tag{5.30}$$

and B^e satisfies the identity

$$C = B^f A_f = 0. \tag{5.31}$$

One can easily check that the terms in the numerator

$$1 - s^2\Sigma_+ + s^4\Sigma_- - s^6 \tag{5.32}$$

in (5.9) are related to the reducibility properties of $\epsilon^{abcde}u_{abc}u_{def} = 0$ in the same manner that the terms in the numerator of (3.16) are related to the reducibility properties of $\lambda\gamma^m\lambda = 0$. The term $-s^2\Sigma_+$ comes from the original A_f constraints, the term $s^4\Sigma_-$ comes from the B^e constraints, and the term $-s^6$ comes from the C constraint.

These constraints can also be obtained from the cohomology of a "mini"-BRST operator, which acts on the space of functions of (u_{abc}, θ_{abc}), where u_{abc} are the components of the constrained bosonic three-form, and θ_{abc} is an unconstrained fermionic three-form. The operator is:

$$q_{brst} = u_{abc}\frac{\partial}{\partial\theta_{abc}} \tag{5.33}$$

and squares to zero. The previous statements (5.29)(5.30)(5.31) translate to the fact that the cohomology of q_{brst} is spanned by:

$$\begin{gathered} 1 \\ \mathcal{A}_f = \varepsilon^{abcde} u_{abc}\theta_{def}, \\ \mathcal{B}^e = \varepsilon^{abcde}\varepsilon^{a'b'c'd'e'}\theta_{aa'b'}u_{c'd'e'}\theta_{bcd}, \\ \mathcal{C} = \varepsilon^{abca'b'}\varepsilon^{defd'e'}\varepsilon^{c'f'a''b''c''}u_{abc}u_{def}\theta_{a'b'c'}\theta_{d'e'f'}\theta_{a''b''c''}. \end{gathered} \tag{5.34}$$

So the numerator of (5.32) is related to the cohomology of (5.33) in the same way that the numerator of (3.16) is related to the cohomology of the $D = 10$ superparticle BRST operator.

5.4. *Computation of $D = 11$ Central Charges*

To covariantly compute the ghost central charge for a $D = 11$ pure spinor, it is convenient to set $\sigma = 0$ in (5.3) to obtain the formula

$$z_{11}(t) = Z_{11}(t,0) = \frac{(1+t)(1+3t+t^2)(1+5t+10t^2+5t^3+t^4)}{(1-t)^{23}}. \tag{5.35}$$

Expansion in $\log(t)$ as in section 4 yields:

$$\boxed{\tfrac{1}{2}c_{\rm Vir} = \sum_n N_n = 23}$$
$$\boxed{a_{\rm ghost} = -\sum_n nN_n = -16} \tag{5.36}$$

The ghost and Lorentz central charges, defined as in the ten dimensional case via the moments of N_n and $N_{n,m}$ turn out to be fractional in eleven dimensions. We don't have any a priori reasons to expect them to have any meaning, so we shall not try to explain their failure to be integers.

6. Twelve-dimensional Pure Spinors

Although the reducibility method for computing the character of twelve-dimensional pure spinors is extremely complicated, the fixed-point method is relatively simple. As will now be shown, the formula for $Z_{2d+2}(t,\sigma)$ is related to $Z_{2d}(t,\sigma)$, which allows one to obtain the $d = 6$ fixed-point formula from the $d = 5$ fixed-point formula of section 3.

6.1. *Recursion relation*

From (3.31) we can derive a recursion formula which relates $Z_{2d}(t,\sigma)$ to $Z_{2d+2}(t,\sigma,0)$ where σ has d components. Thus we can read off the *Spin*$(2d)$ content for the $2(d+1)$ dimensional pure spinors. The special feature of the choice $\sigma_{d+1} = 0$ is that the local contribution (3.31) is non-singular, provided the rest of the angles σ is generic. This would not be the case had we set two angles, say σ_{d+1} and σ_d to zero.

For $\sigma_{d+1} = 0$ the $2(d+1)$-dimensional character takes the form:

$$Z_{2d+2}(t,\sigma,0) = \sum_\epsilon Z_{2d+2;\epsilon} \tag{6.1}$$

where ϵ is the same as in (3.30) (i.e. has d components), but now ($s_\epsilon = \prod_a e^{\frac{i}{2}\epsilon_a\sigma_a}$):

$$Z_{2d+2;\epsilon} \quad = \quad \frac{1}{1-ts_\epsilon^{-1}}\prod_{a=1}^{d}\frac{1}{1-e^{i\epsilon_a\sigma_a}}\prod_{1\le a<b\le d}\frac{1}{1-e^{i(\epsilon_a\sigma_a+\epsilon_b\sigma_b)}} \tag{6.2}$$

$$Z_{2d+2}(t,\sigma,0) = \frac{1}{t\prod_{a=1}^{d}\left(e^{-\frac{i\sigma_a}{2}} - e^{\frac{i\sigma_a}{2}}\right)}\left[Z_{2d}(t,\sigma) - Z_{2d}(t,-\sigma)\right] \tag{6.3}$$

for odd d, and

$$Z_{2d+2}(t,\sigma,0) = \frac{1}{t\prod_{a=1}^{d}\left(e^{-\frac{i\sigma_a}{2}} - e^{\frac{i\sigma_a}{2}}\right)}\left[Z_{2d}(t,\sigma) + Z_{2d}(t,-\sigma) - 2\right] \tag{6.4}$$

for even d.

6.2. On to twelve dimensions

Let us apply the recursion relation (6.3) to the case of twelve dimensional spinors. We get, after some simple arithmetics:

$$Z_{12}(t,\sigma,0) = \frac{P_{12}(t,\sigma)}{Q_{12}(t,\sigma)} \tag{6.5}$$

where

$$\boxed{P_{12}(t,\sigma) \quad = \quad (1-t^2)\left((1+t^2)^3 - t^2(1+t^2)U + \tfrac{1}{2}t^3 S\right)\Gamma - \tfrac{1}{2}t^2(1+t^2)\Theta}$$

$$\boxed{Q_{12}(t,\sigma) \quad = \quad \prod_\epsilon (1-ts_\epsilon) = \mathcal{D}_+\mathcal{D}_-}$$

$$\begin{aligned} & S = S_+ + S_- \ , \\ & U = 2\left(1+\sum_{a=1}^{5}\ \cos\sigma_a\right) \ , \\ & \Theta = \mathcal{D}_+ + \mathcal{D}_- \ , \end{aligned} \tag{6.6}$$

$$\Gamma = -\frac{1}{t}\frac{\mathcal{D}_+ - \mathcal{D}_-}{S_+ - S_-}$$

$$\mathcal{D}_\pm = \prod_{\epsilon,\ \epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5=\pm 1}(1-ts_\epsilon)$$

For $\vec{\sigma}=0$ the character simplifies:

$$z_{12}(t)=Z_{12}(t,0)=\frac{1+16t+70t^2+112t^3+70t^4+16t^5+t^6}{(1-t)^{16}} \tag{6.7}$$

If all but one angle vanish, $\sigma_a=0$, $a=2,\ldots,6$, then the character assumes the form:

$$Z_{12}(t,\sigma_1,0,0,0,0,0)=\frac{\rho_{12}(t,X)}{\left(1-tX+t^2\right)^{11}}$$

$$\begin{aligned}\rho_{12}(t,X)=&\,1-31t^2+187t^4-330t^6+187t^8-31t^{10}+t^{12}+\\&(5-48t^2+55t^4+55t^6-48t^8+5t^{10})tX\\&+(5-11t^2-11t^6+5t^8)t^2X^2+(1+t^6)t^3X^3\end{aligned} \tag{6.8}$$

where $X=2\cos\left(\frac{\sigma_1}{2}\right)$. From (6.7)(6.8) we derive the ghost number anomaly, the ghost number current central charge, Virasoro central charge and the Lorentz current central charge:

$$\boxed{\begin{aligned}&c_{\text{Vir}}=32\\&c_{\text{ghost}}=-4\\&c_{\text{Lorentz}}=-4\\&a_{\text{ghost}}=-10\end{aligned}}\ , \tag{6.9}$$

in agreement with the non-covariant calculation.

6.3. State of the art in twelve dimensions.

We now show the full formula for the twelve dimensional case. The non-trivial part is of course the numerator:

$$P_{12}(t,\sigma)\ =A(t,\sigma)+B(t,\sigma)\ ;$$

$$\begin{aligned}A(t,\sigma)\ =&\,(1+t^{22})-(t^2+t^{20})\,U_{[0,0,0,0,1,1]}-(t^4+t^{18})\,U_{[-1,1,1,1,1,1]}+\\&(t^6+t^{16})\times\left(U_{[0,1,1,1,1,2]}+U_{[0,0,0,0,1,3]}\right)+\\&(t^8+t^{14})\times\left(U_{[0,0,1,1,2,2]}-U_{[0,0,0,0,0,4]}\right)-\\&(t^{10}+t^{12})\times\left(U_{[1,1,1,1,2,2]}+U_{[0,0,0,2,2,2]}+U_{[0,0,1,1,1,3]}\right)\end{aligned} \tag{6.10}$$

$$
\begin{aligned}
B(t,\sigma) \ &= (t^3+t^{19})\, U_{\frac{1}{2}[1,1,1,1,1,3]} - (t^5+t^{17})\, U_{\frac{1}{2}[1,1,1,1,1,5]} - \\
&\quad (t^7+t^{15}) \times \left(U_{\frac{1}{2}[1,1,3,3,3,3]} + U_{\frac{1}{2}[-1,1,1,1,3,5]} \right) \\
&\quad (t^9+t^{13}) \times \left(U_{\frac{1}{2}[3,3,3,3,3,3]} + U_{\frac{1}{2}[-1,1,1,1,1,7]} \right) + 2\, t^{11}\, U_{\frac{1}{2}[1,1,1,3,3,5]}
\end{aligned}
\tag{6.11}
$$

while the denominator is given by the standard product, as in (6.6). We are using the notation U_λ for the character of the irreducible representation of $Spin(12)$ with the highest weight $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]$, where $|\lambda_1| \le \lambda_2 \le \lambda_3 \le \lambda_4 \le \lambda_5 \le \lambda_6$ are integers:

$$
U_\lambda = e^{i(\lambda_1\sigma_1 + \ldots + \lambda_6\sigma_6)} + \ldots \tag{6.12}
$$

For example, $U_{[0,0,0,0,0,0]} = 1$ corresponds to the trivial representation, $U_{[0,0,0,0,0,1]}$ corresponds to the vector $\mathbf{12}$, $U_{\frac{1}{2}[1,1,1,1,1,1]}$ is the chiral spinor S_-, and $U_{\frac{1}{2}[-1,1,1,1,1,1]}$ is the opposite chirality spinor S_+. The dictionary can be continued:

$$
\begin{aligned}
& U_{[0,0,0,0,1,1]} = \Lambda^2 U \\
& U_{[-1,1,1,1,1,1]} = \Lambda^{6,+} U \\
& U_{\frac{1}{2}[1,1,1,1,1,3]} = S_+ U - S_- \\
& U_{\frac{1}{2}[1,1,1,1,1,5]} = S_- \mathrm{Sym}^2 U - S_+ U \\
& U_{\frac{1}{2}[1,1,3,3,3,3]} + U_{\frac{1}{2}[-1,1,1,1,3,5]} = S_- \Lambda^4 U + S_+ U_{[0,0,0,0,1,2]} - \ldots \\
& U_{\frac{1}{2}[3,3,3,3,3,3]} + U_{\frac{1}{2}[-1,1,1,1,1,7]} = S_- \Lambda^{6,+} U + S_+ \mathrm{Sym}^3 U - \ldots \\
& U_{\frac{1}{2}[1,1,1,3,3,5]} = S_- U_{[0,0,0,1,1,2]} - \ldots
\end{aligned}
\tag{6.13}
$$

Weyl formula for the character gives:

$$
U_\lambda = \frac{1}{\Delta_{D_6}} \sum_{\epsilon \in \mathbf{Z}_2^5} \sum_{w \in \mathcal{S}_6} (-)^w e^{iw(\lambda+\rho)\cdot\sigma_\epsilon} \tag{6.14}
$$

where

$$
\begin{aligned}
& \lambda + \rho = [\lambda_1, \lambda_2+1, \lambda_3+2, \lambda_4+3, \lambda_5+4, \lambda_6+5] \\
& w(\mu) = [\mu_{w(1)}, \mu_{w(2)}, \mu_{w(3)}, \mu_{w(4)}, \mu_{w(5)}, \mu_{w(6)}] \\
& \sigma_\epsilon = [\epsilon_1\sigma_1, \epsilon_2\sigma_2, \epsilon_3\sigma_3, \epsilon_4\sigma_4, \epsilon_5\sigma_5, \epsilon_6\sigma_6] \\
& \epsilon = [\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6] \in \mathbf{Z}_2^5, \epsilon_i = \pm 1\,, \prod_{i=1}^{6} \epsilon_i = 1 \\
& \Delta_{D_6} = \sum_{\epsilon \in \mathbf{Z}_2^5} \sum_{w \in \mathcal{S}_6} (-)^w e^{iw(\rho)\cdot\sigma_\epsilon} = \\
& \qquad \prod_{1 \le i < j \le 6} \left(e^{\frac{i(\sigma_i - \sigma_j)}{2}} - e^{\frac{i(\sigma_j - \sigma_i)}{2}} \right) \left(e^{\frac{i(\sigma_i + \sigma_j)}{2}} - e^{-\frac{i(\sigma_j + \sigma_i)}{2}} \right)
\end{aligned}
\tag{6.15}
$$

and the sum in (6.14) is over the elements (ϵ, w) of the Weyl group of $Spin(12)$. The physical interpretation of the field content (6.10)(6.11) is unclear to us. Presumably the middle component, corresponding to the ghost number 11, stands for the pair of a physical field and its antifield. The other components correspond to ghosts, field strengths etc.

The first non-trivial term in (6.10) corresponds to the Q cohomology element $\lambda\Gamma_{MN}\theta$. The degree 3 term corresponds to $(\lambda\Gamma_{MN}\theta)\left(\Gamma^N\theta\right)^\alpha - (\lambda\theta)\left(\Gamma_M\theta\right)^\alpha$. The self-dual six-form at the degree 4 deserves further attention.

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[7] Some of the ideas relating the richness of the cohomology of the pure spinor BRST operator and the reducibility of the quadratic pure spinor constraints were discussed in [15].

[8] The relation of our ten dimensional character formula to the super-Yang-Mills superfields can be viewed as an independent check of the Koszul duality between the (super)-Yang-Mills algebra[16][17] and the algebra of functions on $\mathcal{M}_{10}$, discussed in [18]. Also, M. Movshev has informed us that M. Reed has recently constructed a resolvent of the space of holomorphic functions on $\mathcal{M}_{10}$, from which one can easily deduce our formula (3.14). We take this as a compliment.

Appendix A. On the character calculation in general dimension $2d$

The sum over fixed points can be done explicitly for $D \leq 10$. Once the formula is established, it can be checked by comparing the residues at the points $t = s_\epsilon$.

However, for $D = 12$ the sum over the 2^5 fixed points is very complex, and to bring it to the form (6.10)(6.11) is rather non-trivial. In general the number of fixed points is 2^{d-1} and the complexity of the problem grows exponentially with $N = 2^{d-1}$.

We now present an algorithm, polynomial in N, which gives the t^k contribution to the numerator P_{2d} of the general character formula. The number of operations involved is of the order of $\binom{N}{k}$.

The idea is to recognize in the sum over the fixed points a part of the sum over the elements of the Weyl group of $Spin(2d)$, as in the general formula (6.14). Indeed, the 2^{d-1} fixed points are in one-to-one correspondence with the cosets $W_{Spin(2d)}/W_{U(d)}$, and the denominator in the fixed point formula, apart from the t-dependent factor, is a subfactor in the Weyl denominator Δ_{D_d}. The remaining factor is the Weyl denominator for $U(d)$,

$$\Delta_{A_{d-1}} = \prod_{a<b} (1 - e^{i(\epsilon_a \sigma_a - \epsilon_b \sigma_b)})$$

which can be introduced at the expense of extending the sum over the cosets to the sum over all of the Weyl group of $Spin(2d)$.

These considerations lead to the following prescription. Consider the Cartan subalgebra $\mathbf{h}_d$ of $Spin(2d)$. It is a d-dimensional vector space. Let $\mathbf{e}_a$, $a = 1, \ldots, d$, denote the orthonormal basis in $\mathbf{h}_d$. In our previous notations:

$$\mathbf{e}_a = [000 \ldots 1 \ldots 0]$$

(1 on the a'th spot). The positive roots of $Spin(2d)$ are: $\mathbf{e}_a - \mathbf{e}_b$, $\mathbf{e}_a + \mathbf{e}_b$, $1 \leq a < b \leq d$. Let ρ denote the half of the sum of the positive roots of $Spin(2d)$:

$$\rho = \sum_{a=1}^{d} (a-1)\mathbf{e}_a \ ,$$

Let S_-, as before, denote the character of the chiral spinor representation, the one where $\theta_\alpha, \lambda_\alpha$ take values:

$$S_- = \sum_{\epsilon \in \mathbf{Z}_2^{d-1}} e^{\frac{i}{2}\epsilon \cdot \sigma}$$

We can expand the weights $\mu = \frac{1}{2}\epsilon$ in the basis $\mathbf{e}_a$:

$$\mu = \tfrac{1}{2}\sum_a \epsilon_a \mathbf{e}_a \ . \tag{A.1}$$

Let us now introduce an ordering on the weights μ:

$$\mu_1 > \mu_2 \quad \text{iff} \quad \mu_1 \cdot \mathbf{e}_a = \mu_2 \cdot \mathbf{e}_a, \ a = 1, \ldots \ \ell - 1, \qquad \mu_1 \cdot \mathbf{e}_\ell > \mu_2 \cdot \mathbf{e}_\ell$$

Thus:

$$\tfrac{1}{2}[+1+1\ldots+1] > \tfrac{1}{2}[-1+1+1\ldots+1] > \ldots \tfrac{1}{2}[-1-1\ldots-1] \tag{A.2}$$

Let $\lambda_0 = \frac{1}{2}[+1+1\ldots+1]$ - the maximal (highest) weight.

Then the t^k contribution to P_{2d} is given by $(-1)^k$ times the sum of the characters of the irreducible highest weight representations U_λ, with multiplicity $m_\lambda^{(k)}$, where λ obeys: $\lambda + \rho = w(\mu_1 + \ldots + \mu_k + \rho)$, for some $\lambda_0 > \mu_k > \ldots \mu_1$, ($\lambda_0$ is absent because it is cancelled by the t-dependent factor in the denominator of the fixed point formula), w is an element of the Weyl group $W_{D_d} = W_{A_{d-1}} \ltimes \mathbf{Z}_2^{d-1}$ which makes $\lambda = [\lambda_1, \ldots, \lambda_d]$ a dominant weight (i.e. $|\lambda_1| \leq \lambda_2 \leq \ldots \leq \lambda_d$). Finally, if such a w exists, then it is unique, and the set $\mu_1, \ldots, \mu_k$ contributes to the multiplicity $m_\lambda^{(k)}$ the amount $(-1)^w$, which is the signature of the permutation π_w in $W_{A_{d-1}} \equiv \mathcal{S}_d$ (w is a composition of the permutation π_w and the flip of an even number of signs), otherwise it contributes 0 to $m_\lambda^{(k)}$. This gives a prescription for computing

$$\begin{aligned} P_{2d} &= \sum_{k=0}^{2^{d-1}-2d+2} (-t)^k \sum_\lambda m_\lambda^{(k)} U_\lambda \\ m_\lambda^{(k)} &= \sum_{\mu_1 < \ldots < \mu_k < \lambda_0, w, \lambda+\rho = w(\mu_1+\ldots+\mu_k+\rho)} (-1)^{\pi_w} \end{aligned} \tag{A.3}$$

The sum in (A.3) goes over the k-tuples of weights from (A.1), except for λ_0, ordered according to (A.2), and over the elements w in the Weyl group of $Spin(2d)$, such that the application of w to $\mu_1 + \ldots + \mu_k + \rho$ gives $\lambda + \rho$.

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